BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Measure Theory – Part 16

Proof of the substitution rule for measure spaces

$h: X \to Y$ measurable , $g: Y \to \mathbb{R}$

$$\int_Y g \, d(h_* \mu) = \int_X g \circ h \, d\mu$$

Proof: (1) Let $g = \chi_C$ with $C \subseteq Y$ measurable

$$\int_Y \chi_C \, d(h_*\mu) = (h_*\mu)(C) = \mu(h^{-1}(C))$$

$$\int_X \chi_C \circ h \, d\mu = \int_X \underbrace{\chi_C(h(x))}_{= \begin{cases} 1 , & x \in h^{-1}(C) \\ 0 , & x \notin h^{-1}(C) \end{cases}} d\mu(x) = \int_X \chi_{h^{-1}(C)} \, d\mu \quad (\parallel)$$

(2) Let g be a simple function, i.e. $g = \sum_{i=1}^n \lambda_i \chi_{C_i}$

$$\int_Y \sum_{i=1}^n \lambda_i \chi_{C_i} \, d(h_*\mu) = \sum_{i=1}^n \lambda_i \int_Y \chi_{C_i} \, d(h_*\mu) \overset{(1)}{=} \sum_{i=1}^n \lambda_i \int_X \chi_{C_i}(h(x)) \, d\mu(x)$$

$$= \int_X \Big(\sum_{i=1}^n \lambda_i \chi_{C_i} \Big)(h(x)) \, d\mu(x)$$

$$= \int_X g \circ h \, d\mu$$

(3) Let $g: Y \to [0, \infty)$ measurable

$$\int_Y g \, d(h_*\mu) = \sup \Big\{ \int_Y \tilde{s} \, d(h_*\mu) \;\Big|\; \tilde{s}: Y \to [0,\infty) \text{ simple} , \tilde{s} \leq g \Big\}$$

$\forall y \in h(X): \tilde{s}(y) \leq g(y)$

$\Leftrightarrow \forall x \in X : \underbrace{\tilde{s}(h(x))}_{\tilde{s} \circ h} \leq \underbrace{g(h(x))}_{(g \circ h)(x)}$

$\tilde{s} \circ h$ ← simple function on X

$$= \sup \Big\{ \int_X \tilde{s} \circ h \, d\mu \;\Big|\; \tilde{s}: Y \to [0,\infty) \text{ simple} , . \; \tilde{s} \circ h \leq g \circ h \Big\}$$

$$\overset{(*)}{=} \sup \Big\{ \int_X s \, d\mu \;\Big|\; s: X \to [0,\infty) \text{ simple} , \; s \leq g \circ h \Big\}$$

$$= \int_X g \circ h \, d\mu$$

For understanding $(*)$, let's do a quick sketch: One inequality is clear $(\leq)$ ✓

To prove $(\geq)$ we need to show: For given $s: X \to [0,\infty)$ with $s \leq g \circ h$

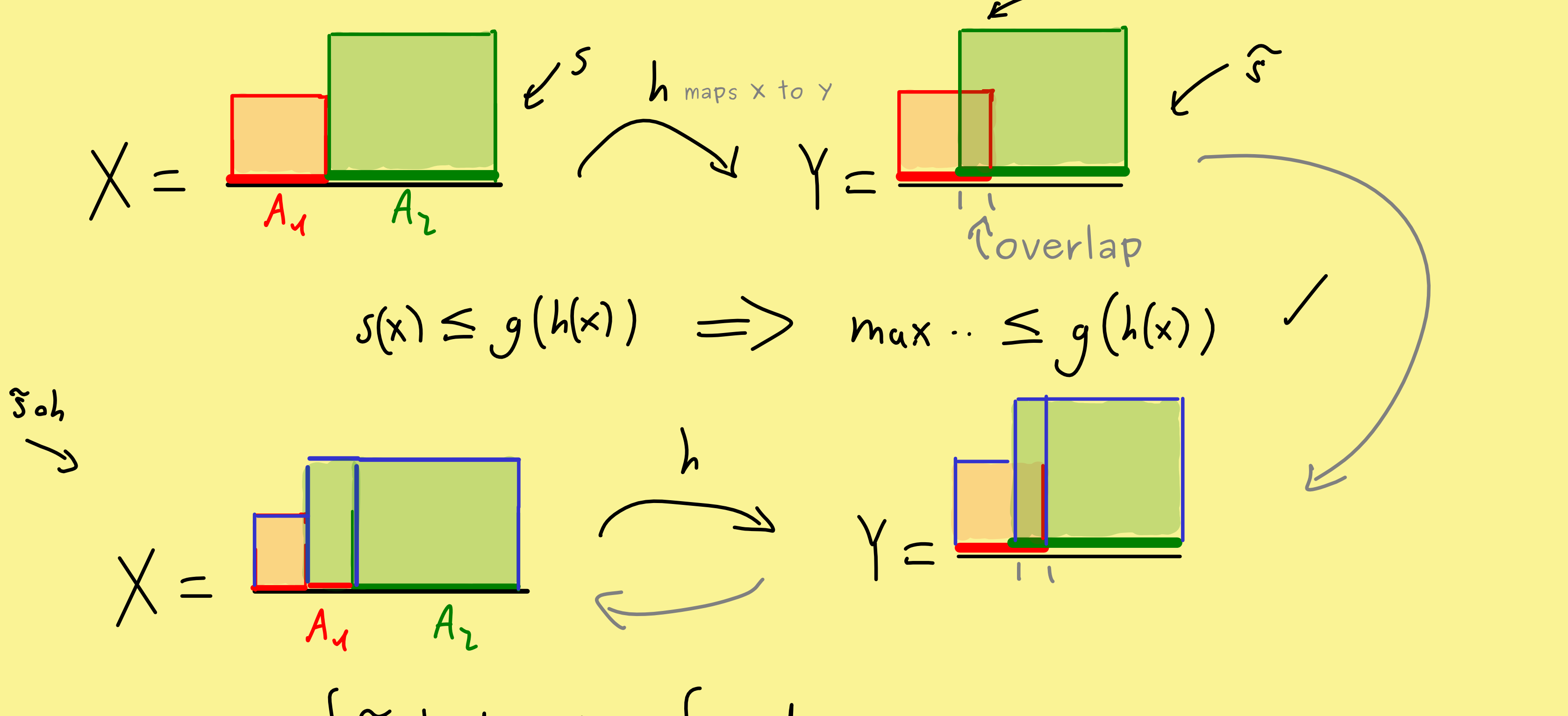
find $\tilde{s}: Y \to [0, \infty)$ with $\tilde{s} \leq g \circ h$

and $\int \tilde{s} \circ h \geq \int s$

A problem only occurs if h is not injective!

Let s be given as $s(x) = \sum_{i=1}^n a_i \chi_{A_i}(x)$ with pairwise disjoint A_i

A sketch should be sufficient:



Hence: $\int \tilde{s} \circ h \, d\mu \geq \int s \, d\mu$