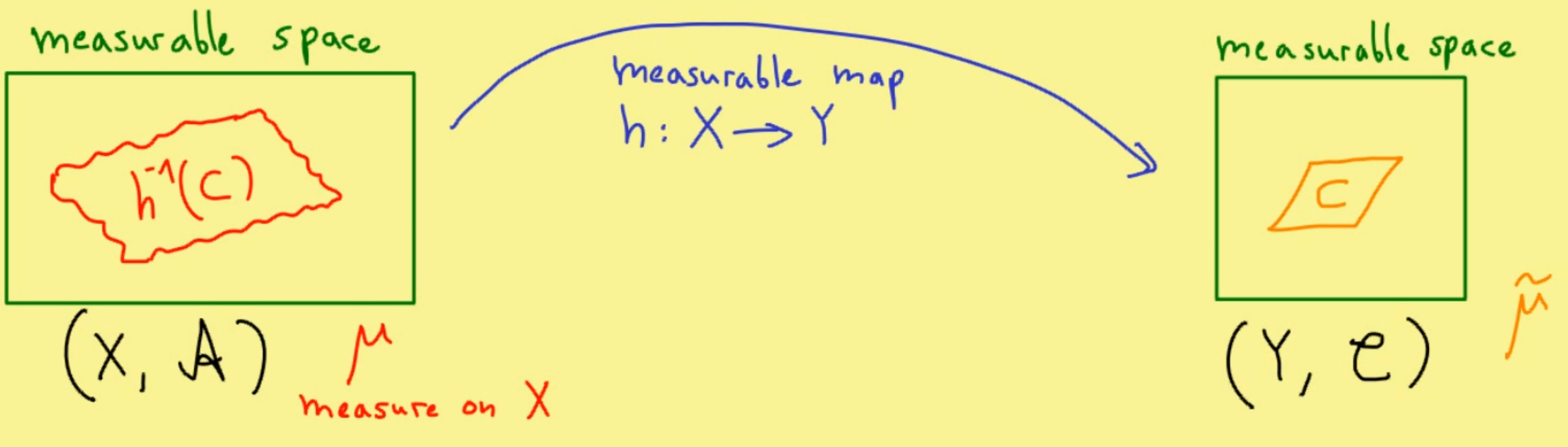
The Bright Side of Mathematics

Measure Theory – Part 15

Image measure and substitution rule

↳ push forward measure ↳ change of variables



$\tilde{\mu}(C) = \mu(h^{-1}(C))$ Image measure.

Notations: $h_* \mu$ or $\mu \circ h^{-1}$ (measure on Y)

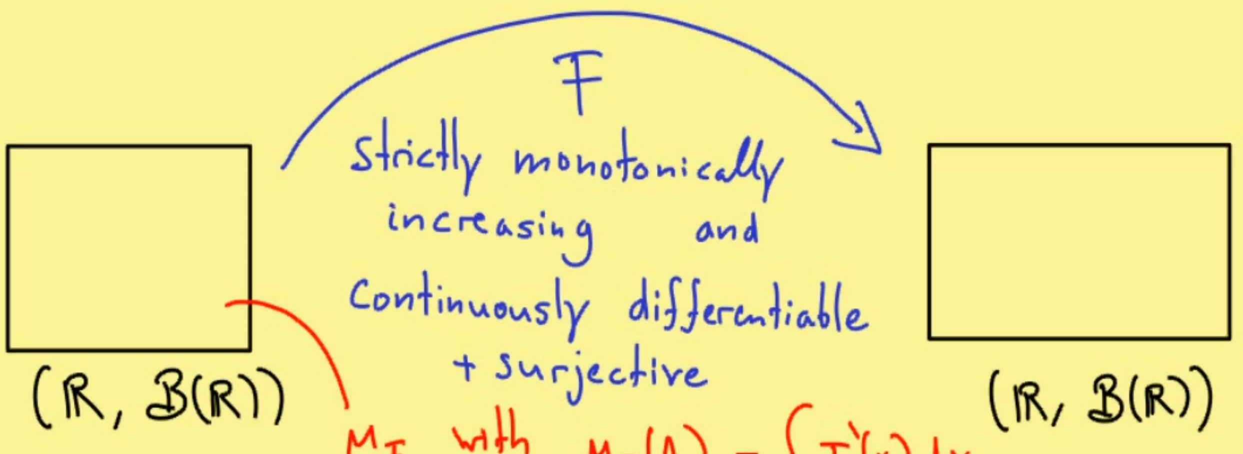
Substitution formula: $g: Y \to \mathbb{R}$

$$\int_Y g \, d(h_* \mu) = \int_X g \circ h \, d\mu$$

$$\int_Y g(y) \, d(\mu \circ h^{-1})(y) = \int_X g(h(x)) \, d\mu(x)$$

Change of variables: $y = h(x)$

Example:



μ_F with $\mu_F(A) = \int_A F'(x)\,dx$

$$(F_* \mu_F)([a,b)) = \mu_F(F^{-1}([a,b))) = \mu_F([F^{-1}(a), F^{-1}(b)))$$

$$= \int_{F^{-1}(a)}^{F^{-1}(b)} F'(x)\,dx = \int_a^b dy$$

$y = F(x)$
$dy = F'(x)\,dx$

$$= \lambda([a,b)) \Rightarrow F_* \mu_F = \lambda$$

Substitution formula:

$$\int_Y g \, d(F_* \mu_F) = \int_X g \circ F \, d\mu_F$$

$$\int_{\mathbb{R}} g(y)\,dy = \int_{\mathbb{R}} g(F(x))\, F'(x)\,dx$$