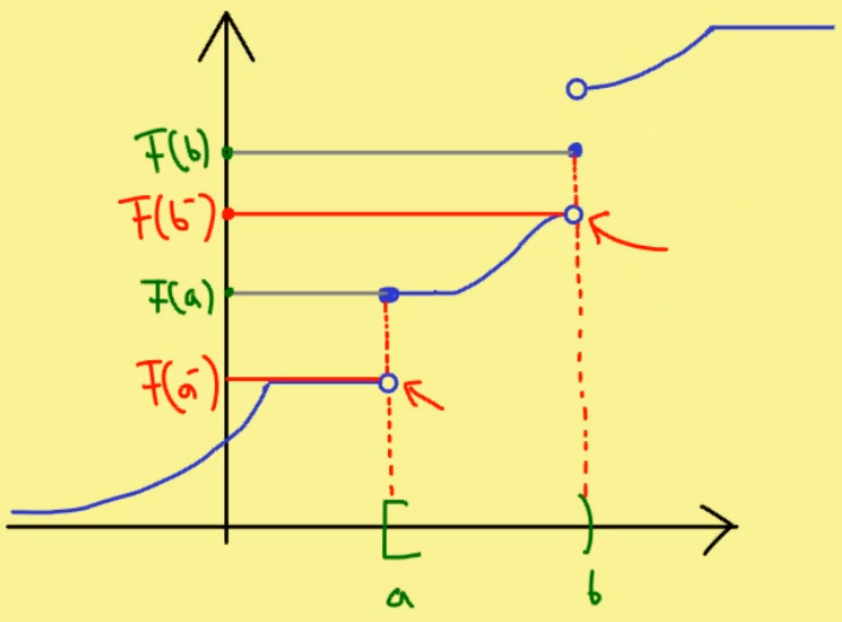
Measure Theory – Part 13

Lebesgue-Stieltjes measures

$F: \mathbb{R} \to \mathbb{R}$ monotonically increasing (non-decreasing)



length of interval $[a,b)$:

$$F(b^-) - F(a^-) =: \mu_F([a,b))$$

where $F(a^-) := \lim_{\varepsilon \to 0^+} F(a-\varepsilon)$

Alternative $F(b^+) - F(a^+) = \mu_F((a,b])$

where $F(a^+) := \lim_{\varepsilon \to 0^+} F(a+\varepsilon)$

$\mathcal{A} := \{ [a,b) : a,b \in \mathbb{R},\ a \leq b \}$ semiring of sets

$\xRightarrow{\text{Carathéodory}}$ There exists exactly one measure $\mu_F : \mathcal{B}(\mathbb{R}) \to [0,\infty]$ with $(*)$

It's called Lebesgue-Stieltjes measure for F

Examples:

(a) $F(x) = x$, $\mu_F([a,b)) = b-a \rightsquigarrow$ Lebesgue measure

(b) $F(x) = 1$, $\mu_F([a,b)) = 0 \rightsquigarrow$ Zero measure

(c) $F(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$ $\quad \mu_F([-\varepsilon,\varepsilon)) = 1 \rightsquigarrow$ Dirac measure δ_0

(d) $F: \mathbb{R} \to \mathbb{R}$ monotonically increasing + continuously differentiable

$F': \mathbb{R} \to [0,\infty)$. $\quad \mu_F([a,b)) = F(b) - F(a) = \int_a^b F'(x)\,dx$

$$\rightsquigarrow \quad \mu_F : A \mapsto \int_A \underbrace{F'(x)}_{\text{density function}} dx$$