The Bright Side of Mathematics

Measure Theory – Part 12

Carathéodory's extension theorem

X set , $\mathcal{A} \subseteq \mathcal{P}(X)$ semiring of sets

$\mu : \mathcal{A} \longrightarrow [0,\infty]$ pre-measure

(a) Then μ has an extension

measure $\rightarrow$ $\tilde{\mu} : \sigma(\mathcal{A}) \longrightarrow [0,\infty]$, i.e. $\mu(A) = \tilde{\mu}(A)$ for all $A \in \mathcal{A}$

(b) If there is sequence (S_j) with $S_j \in \mathcal{A}$, $\bigcup_{j=1}^{\infty} S_j = X$, $\mu(S_j) < \infty$, then the extension $\tilde{\mu}$ from (a) is unique. ($\tilde{\mu}$ is also σ-finite)

Explanations Semiring of sets $\mathcal{A} \subseteq \mathcal{P}(X)$

(1) $\emptyset \in \mathcal{A}$ (as for σ-algebras)

(2) $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$

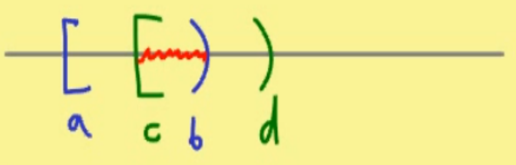
(3) For $A, B \in \mathcal{A}$, there are pairwise disjoint sets $S_1, \dots, S_n \in \mathcal{A}$:
$$\bigcup_{j=1}^{n} S_j = A \setminus B$$

Most important example: $\mathcal{A} := \{ [a,b) \mid a, b \in \mathbb{R},\ a \le b \}$

not a σ-algebra because $\mathbb{R} \notin \mathcal{A}$.

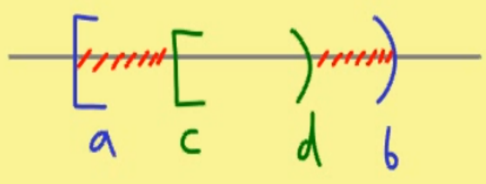
but $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$ (Borel σ-algebra)

(1) $\emptyset \in \mathcal{A}$ ✓



(2) $[a,b) \cap [c,d) = \begin{cases} \emptyset, & b \le c,\ d \le a \\ [c,b), & c \in [a,b) \not\ni d \\ \vdots & \end{cases}$ ✓

(3) $[a,b) \setminus [c,d) = \begin{cases} [a,b), & b \le c,\ d \le a \\ [a,c), & c \in [a,b) \not\ni d \\ [a,c) \cup [d,b) & \end{cases}$ ✓.



Pre-measure: $\mu : \mathcal{A} \longrightarrow [0,\infty]$ with $\mathcal{A}$ semiring of sets:

(a) $\mu(\emptyset) = 0$

(b) $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$, for $A_j \in \mathcal{A}$, $A_i \cap A_j = \emptyset$ for $i \ne j$

and $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$.

Application: $\mathcal{A} := \{ [a,b) \mid a, b \in \mathbb{R},\ a \le b \}$ semiring of sets

$\mu : \mathcal{A} \longrightarrow [0,\infty]$, $\mu([a,b)) = b - a$ pre-measure

$\overset{\text{Carathéodory}}{\Longrightarrow}$ There is a unique extension to $\mathcal{B}(\mathbb{R})$ $\Rightarrow$ $\boxed{\text{Lebesgue measure}}$