The Bright Side of Mathematics

Measure Theory – Part 11

Proof of Lebesgue's dominated convergence theorem

Lebesgue's dominated convergence theorem

$f_n : X \rightarrow \mathbb{R}$ measurable for all $n \in \mathbb{N}$

$f : X \rightarrow \mathbb{R}$ with $f_n(x) \xrightarrow{n\to\infty} f(x)$ for $x \in X$ (μ-a.e.)

and $|f_n| \leq g$ with $g \in \mathcal{L}^1(\mu)$ for all $n \in \mathbb{N}$.

integrable majorant

Then: $f_1, f_2, f_3, \ldots \in \mathcal{L}^1(\mu)$, $f \in \mathcal{L}^1(\mu)$

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu .$$

Proof: $|f_n| \leq g \overset{\text{monotonicity}}{\Longrightarrow} \int_X |f_n| \, d\mu \leq \int_X g \, d\mu < \infty$

$\Rightarrow f_1, f_2, f_3, \ldots \in \mathcal{L}^1(\mu)$

$|f| \leq g$ μ-a.e. $\overset{\text{monot.}}{\Longrightarrow} f \in \mathcal{L}^1(\mu)$

We will show: $\int_X |f_n - f| \, d\mu \xrightarrow{n\to\infty} 0$

$|f_n - f| \leq |f_n| + |f| \leq 2g \Rightarrow h_n := 2g - |f_n - f| \geq 0$

Hence: $h_n : X \rightarrow [0,\infty]$ measurable for all $n \in \mathbb{N}$

$$\overset{\text{Fatou}}{\Longrightarrow} \int_X \liminf_{n\to\infty} h_n \, d\mu \leq \liminf_{n\to\infty} \int_X h_n \, d\mu$$

$$\int_X \liminf_{n\to\infty} h_n \, d\mu = \int_X 2g \, d\mu$$

$$\liminf_{n\to\infty} \int_X h_n \, d\mu = \int_X 2g \, d\mu - \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu$$

$$\Rightarrow \quad 0 \leq \liminf_{n\to\infty} \int_X |f_n - f| \, d\mu \leq \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu \leq 0$$

$\Rightarrow$ limit exists and $\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0$.

We conclude:

$$0 \leq \left| \int_X f_n \, d\mu - \int_X f \, d\mu \right| = \left| \int_X (f_n - f) \, d\mu \right|$$

$$\leq \int_X |f_n - f| \, d\mu \xrightarrow{n\to\infty} 0$$

$$\Rightarrow \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu \qquad \square$$