The Bright Side of Mathematics

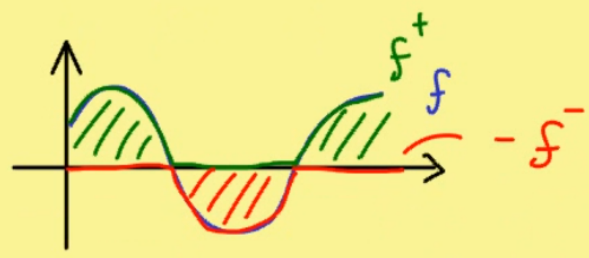
Measure Theory – Part 10

Lebesgue's dominated convergence theorem

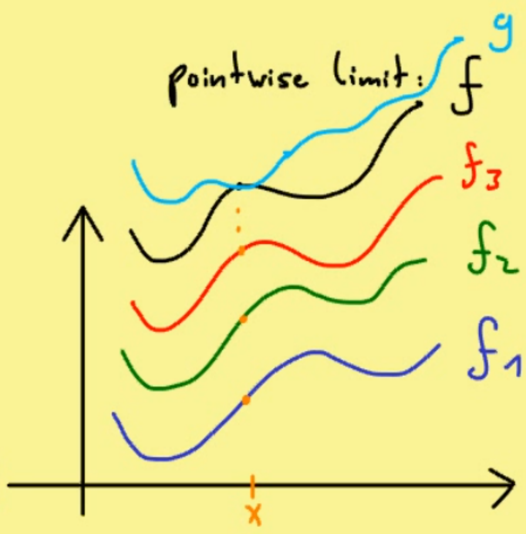
$(X, \mathcal{A}, \mu)$, $\mathcal{L}^1(\mu) := \{ f: X \to \mathbb{R} \text{ measurable} \mid \int_X |f|^1 \, d\mu < \infty \}$

For $f \in \mathcal{L}^1(\mu)$, write $f = f^+ - f^-$, $f^+, f^- \geq 0$

Define: $\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^- \, d\mu$



Lebesgue's dominated convergence theorem



$f_n : X \to \mathbb{R}$ measurable for all $n \in \mathbb{N}$

$f : X \to \mathbb{R}$ with:

$$f_n(x) \xrightarrow{n \to \infty} f(x) \quad \text{for } x \in X \quad (\mu\text{-a.e.})$$

and $|f_n| \leq g$ with $g \in \mathcal{L}^1(\mu)$ for all $n \in \mathbb{N}$.

integrable majorant

Then: $f_1, f_2, f_3, \ldots \in \mathcal{L}^1(\mu)$, $f \in \mathcal{L}^1(\mu)$

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu .$$