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ON STEADY

The Bright Side of Mathematics

Measure Theory – Part 8

Monotone convergence theorem

$(X, \mathcal{A}, \mu)$ measure space, $f_n : X \to [0,\infty]$ $\big(f: X \to [0,\infty] \big)$

measurable for all $n \in \mathbb{N}$

with $f_1 \leq f_2 \leq f_3 \leq \cdots$ μ-a.e.

$\big(\lim_{n\to\infty} f_n(x) = f(x) \quad \mu\text{-a.e. } (x \in X) \big)$

$$\Rightarrow \quad \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \overbrace{\lim_{n\to\infty} f_n}^{f} \, d\mu \, .$$

Proof:

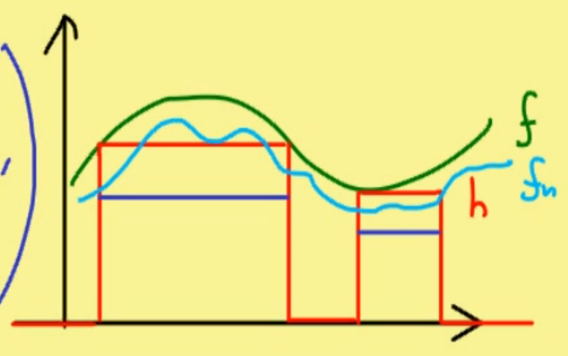
$$\int_X f_1 \, d\mu \leq \int_X f_2 \, d\mu \leq \int_X f_3 \, d\mu \leq \cdots$$

$$\int_X f_n \, d\mu \leq \int_X f \, d\mu \quad \text{for } n \in \mathbb{N}$$

$$\Rightarrow \quad \lim_{n\to\infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu \qquad \big(\text{One part of } (*)\big)$$

Let h be a simple function $0 \leq h \leq f$ and $\varepsilon > 0$.

$$X_n := \{ x \in X \mid f_n(x) \geq (1-\varepsilon) h(x) \} \qquad \left(\begin{array}{l} \bigcup_{n=1}^{\infty} X_n = \tilde{X}, \\ \mu(\tilde{X}^c) = 0 \end{array} \right)$$



$$\int_X f_n \, d\mu \geq \int_{X_n} f_n \, d\mu \geq \int_{X_n} (1-\varepsilon) h \, d\mu$$

$$\lim_{n\to\infty} \int_X f_n \, d\mu \geq \lim_{n\to\infty} \int_{X_n} (1-\varepsilon) h \, d\mu \overset{\text{details for you}}{=} \int_{\tilde{X}} (1-\varepsilon) h \, d\mu = \int_X (1-\varepsilon) h \, d\mu$$

$$\overset{\varepsilon > 0 \text{ arbitrary}}{\Longrightarrow} \quad \lim_{n\to\infty} \int_X f_n \, d\mu \geq \int_X h \, d\mu$$

$$\underset{h \leq f}{\overset{h \text{ arbitrary}}{\Longrightarrow}} \quad \lim_{n\to\infty} \int_X f_n \, d\mu \geq \int_X f \, d\mu \qquad \big(\text{Second part of } (*)\big) \quad \square$$

Application: $(g_n)_{n\in\mathbb{N}}$, $g_n : X \to [0,\infty]$ measurable for all n.

$$\Rightarrow \quad \sum_{n=1}^{\infty} g_n : X \to [0,\infty] \quad \text{measurable}$$

$$\int_X \sum_{n=1}^{\infty} g_n \, d\mu = \sum_{n=1}^{\infty} \int_X g_n \, d\mu$$