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ON STEADY

The Bright Side of Mathematics

Measure Theory – Part 7

$f : X \to [0, \infty)$ measurable

$$\Rightarrow \int_X f \, d\mu \in [0, \infty]$$

Properties: $f, g : X \to [0, \infty)$ measurable

(a) $f = g$ μ-a.e. $\Rightarrow \int_X f \, d\mu = \int_X g \, d\mu$

μ-almost everywhere

$$\mu(\{x \in X \mid f(x) \neq g(x)\}) = 0$$

(b) $f \leq g$ μ-a.e. $\Rightarrow \int_X f \, d\mu \leq \int_X g \, d\mu$

(c) $f = 0$ μ-a.e. $\Leftrightarrow \int_X f \, d\mu = 0$

Proof for (b): Let $h : X \to [0, \infty)$ be a simple function,

i.e. $h(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x) = \sum_{t \in h(X)} t \cdot \chi_{\{x \in X \mid h(x) = t\}}$.

Let $X = \tilde{X}^c \cup \tilde{X}$ with $\mu(\tilde{X}^c) = 0$.

$$\tilde{h}(x) := \begin{cases} h(x), & x \in \tilde{X} \\ a, & x \in \tilde{X}^c \end{cases}, \quad \tilde{h}(x) = \sum_{t \in h(X)} t \cdot \chi_{\{x \in \tilde{X} \mid h(x) = t\}} + a \cdot \chi_{\tilde{X}^c}$$

$$I(\tilde{h}) = \sum_{t \in h(X)} t \cdot \mu(\{x \in \tilde{X} \mid h(x) = t\}) + \underbrace{a \cdot \mu(\tilde{X}^c)}_{= 0}$$

$$= \sum_{t \in h(X)} t \left(\mu(\{x \in \tilde{X} \mid h(x) = t\}) + \underbrace{\mu(\{x \in \tilde{X}^c \mid h(x) = t\})}_{= 0} \right)$$

$$I(h) = \sum_{t \in h(X) \setminus \{0\}} t \cdot \mu(\{x \in X \mid h(x) = t\})$$

$$\tilde{X} := \{x \in X \mid f(x) \leq g(x)\}, \quad \mu(\tilde{X}^c) = 0$$

$$\int_X f \, d\mu = \sup\{I(h) \mid h \in S^+, h \leq f\} = \sup\{I(\tilde{h}) \mid \tilde{h} \in S^+, \tilde{h} \leq f \text{ on } \tilde{X}\}$$

$\leq g$

$$\leq \sup\{I(\tilde{h}) \mid \tilde{h} \in S^+, \tilde{h} \leq g \text{ on } \tilde{X}\}$$

$$= \sup\{I(h) \mid h \in S^+, h \leq g\} = \int_X g \, d\mu$$

Monotone convergence theorem

$(X, \mathcal{A}, \mu)$ measure space, $f_n : X \to [0, \infty)$, $f : X \to [0, \infty)$ measurable for all $n \in \mathbb{N}$

with $f_1 \leq f_2 \leq f_3 \leq \cdots$ μ-a.e.

$\lim_{n \to \infty} f_n(x) = f(x)$ μ-a.e. $(x \in X)$

$$\Rightarrow \lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$