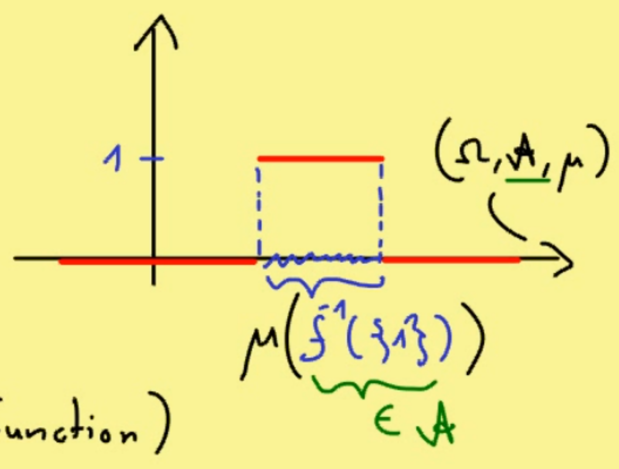
The Bright Side of Mathematics

Measure Theory – Part 5

Definition: $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$ measurable spaces

$f: \Omega_1 \to \Omega_2$ measurable (w.r.t. $\mathcal{A}_1, \mathcal{A}_2$)

if $f^{-1}(A_2) \in \mathcal{A}_1$ for all $A_2 \in \mathcal{A}_2$



Examples: (1) $(\Omega, \mathcal{A})$, $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Characteristic function (also indicator function)

$$\chi_A : \Omega \to \mathbb{R}, \quad \chi_A(\omega) := \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

For all measurable $A \in \mathcal{A}$,

χ_A is a measurable map.

$$\underbrace{\chi_A^{-1}(\emptyset) = \emptyset, \quad \chi_A^{-1}(\mathbb{R}) = \Omega}_{\in \mathcal{A}}$$

$$\underbrace{\chi_A^{-1}(\{1\}) = A, \quad \chi_A^{-1}(\{0\}) = A^c}_{\in \mathcal{A}}$$

(2) $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$, $(\Omega_3, \mathcal{A}_3)$ measurable spaces

$$\Omega_1 \xrightarrow{f} \Omega_2 \xrightarrow{g} \Omega_3, \qquad g \circ f : \Omega_1 \to \Omega_3$$

$$\boxed{f, g \text{ measurable} \Rightarrow g \circ f \text{ measurable}}$$

$$(g \circ f)^{-1}(A_3) = \underbrace{f^{-1}(\underbrace{g^{-1}(A_3)}_{\in \mathcal{A}_2})}_{\in \mathcal{A}_1}$$

Important: $(\Omega, \mathcal{A})$, $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$f, g : \Omega \to \mathbb{R}$ measurable $\Rightarrow$ $f+g$, $f-g$ measurable

$f \cdot g$ measurable

$|f|$ measurable