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The Bright Side of Mathematics

# Measure Theory – Part 4

Measure problem: Search measure $\mu$ on $\mathcal{P}(\mathbb{R})$ with:

(1) $\mu([a,b]) = b-a \quad , \quad b > a.$

(2) $\mu(x+A) = \mu(A) \quad , \quad A \in \mathcal{P}(\mathbb{R}) \, , \, x \in \mathbb{R}.$

$\Rightarrow$ $\mu$ does not exist.

Claim: Let $\mu$ be a measure on $\mathcal{P}(\mathbb{R})$ with $\mu((0,1]) < \infty$ and (2). $\Rightarrow$ $\mu = 0$.

Proof: (a) Definitions: $I := (0,1]$ with equivalence relation on $I$:

$$x \sim y \quad :\Leftrightarrow \quad x - y \in \mathbb{Q}$$

i.e. $[x] := \{ x + r \mid r \in \mathbb{Q} \, , \, x + r \in I \}$

$[x_1] \; [x_2] \; [x_3] \; [x_4] \; \ldots\ldots = I$

$\{a_1, a_2, a_3, a_4 \ldots\} = A$

[disjoint decomposition of $I$ into boxes, possibly uncountable many of them!]

$A \subseteq I$ with property:

(i) For each $[x]$, there is an $a \in A$ with $a \in [x]$.

(ii) For all $a,b \in A$: $a,b \in [x] \Rightarrow a = b$.

[We need axiom of choice of set theory]

$A_n := r_n + A$, where $(r_n)_{n\in\mathbb{N}}$ enumeration of $\mathbb{Q} \cap (-1,1]$.

(b) Claim: $A_n \cap A_m = \emptyset \Leftarrow n \neq m$.

Proof: $x \in A_n \cap A_m \Rightarrow$ $x = r_n + a_n \, , \, a_n \in A$; $x = r_m + a_m \, , \, a_m \in A$

$\Rightarrow r_n + a_n = r_m + a_m \Rightarrow a_n - a_m = r_m - r_n \in \mathbb{Q} \Rightarrow a_n \sim a_m$

$\Rightarrow a_m, a_n \in [a_m] \Rightarrow a_n = a_m \Rightarrow r_n = r_m \Rightarrow n = m.$

(c) Claim: $(0,1] \subseteq \bigcup_{n\in\mathbb{N}} A_n \subseteq (-1,2]$ Proof: Exercise for you!

Assume now: $\mu$ measure on $\mathcal{P}(\mathbb{R})$ with $\mu((0,1]) < \infty$ and (2).

By (2): $\mu(\underbrace{r_n + A}_{A_n}) = \mu(A)$ for all $n \in \mathbb{N}$.

By (c): $\boxed{\mu((0,1]) \leq \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \mu((-1,2])}$. $(*)$

We know: $\mu((0,1]) =: C < \infty$

$\mu((-1,2]) = \mu((-1,0] \cup (0,1] \cup (1,2]) = 3C$ (by using (2) and $\sigma$-add.)

$\overset{(*),(b)}{\Rightarrow} C \leq \sum_{n=1}^{\infty} \mu(A_n) \leq 3 \cdot C \Rightarrow \boxed{C \leq \sum_{n=1}^{\infty} \mu(A) \leq 3 \cdot C}$

$\Rightarrow \mu(A) = 0. \Rightarrow C = 0$ (hence: $\mu((0,1]) = 0$)

$\Rightarrow \mu(\mathbb{R}) = \mu\left(\bigcup_{m\in\mathbb{Z}} (m, m+1]\right) = 0 \Rightarrow \mu = 0. \; \square$