The Bright Side of Mathematics

Measure Theory – Part 3

What is a measure?

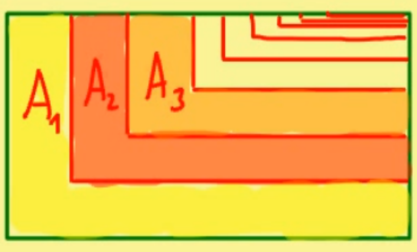
Definition: $(X, \mathcal{A})$ measurable space.

A map $\mu : \mathcal{A} \to [0, \infty]$ $:= [0,\infty) \cup \{\infty\}$ is called a measure if it satisfies:

(a) $\mu(\emptyset) = 0$

σ-additive: (b) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ with $A_i \cap A_j = \emptyset$, $i \neq j$

for all $A_i \in \mathcal{A}$.



Sequence $(A_1, A_2, A_3, \dots)$

$(X, \mathcal{A}, \mu)$ measure space.

Examples: X, $\mathcal{A} = P(X)$

(a) Counting measure: $(A \in \mathcal{A})$

$$\mu(A) := \begin{cases} \#A & , A \text{ has finitely many elements} \\ \infty & , \text{else} \end{cases}$$

number of elements in A

Calculation rules in $[0,\infty]$:

$x + \infty := \infty$ for all $x \in [0,\infty]$

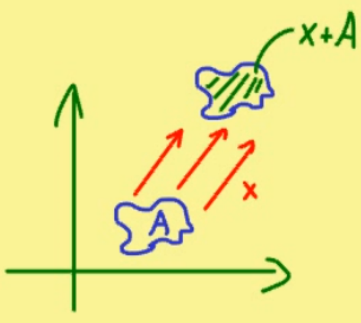
$x \cdot \infty := \infty$ for all $x \in (0,\infty]$

$0 \cdot \infty := 0$ (! in most cases in measure theory!)

(b) Dirac measure for $p \in X$

$$\delta_p(A) := \begin{cases} 1 & , p \in A \\ 0 & , \text{else} \end{cases}$$

(c) We search a measure on $X = \mathbb{R}^n$:



Lebesgue measure (σ-algebra $\neq$ power set)

(1) $\mu([0,1]^n) = 1$

(2) $\mu(x + A) = \mu(A)$ for all $x \in \mathbb{R}^n$