The Bright Side of Mathematics

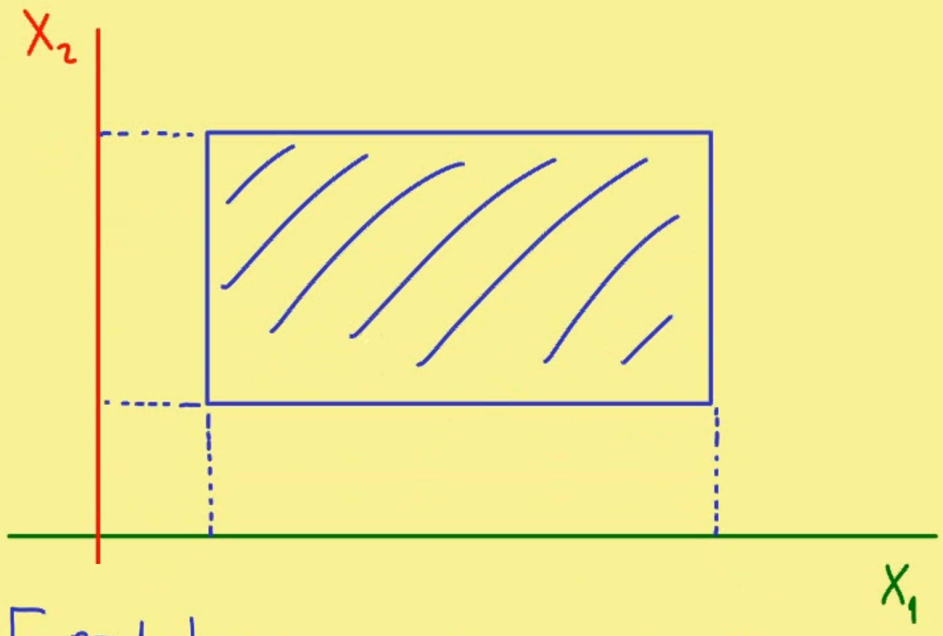
Measure Theory – Part 19

Fubini's Theorem

$(X_1, \mathcal{A}_1, \mu_1)$ measure space

$(X_2, \mathcal{A}_2, \mu_2)$ measure space

$\rightsquigarrow (X_1 \times X_2, \mathcal{A}, \mu)$ product measure



$$\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$$

[product measure is uniquely determined if μ_1, μ_2 are σ-finite]

Fubini's theorem: Let μ_1, μ_2 be σ-finite, μ be the product measure and

$f: X_1 \times X_2 \to [0, \infty]$ measurable. [or $f \in \mathcal{L}^1(\mu)$]

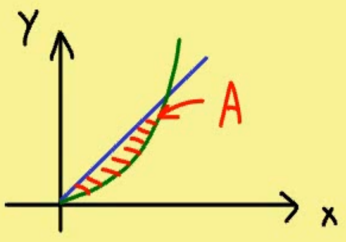
Then:
$$\int_{X_1 \times X_2} f \, d\mu = \int_{X_2} \left(\int_{X_1} f(x,y) \, d\mu_1(x) \right) d\mu_2(y) = \int_{X_1} \left(\int_{X_2} f(x,y) \, d\mu_2(y) \right) d\mu_1(x)$$

Example: μ Lebesgue measure for $\mathbb{R}^2$

$$\int_A f \, d\mu = ?$$

$A = \{ (x,y) \in [0,1] \times [0,1] \mid \underline{x} \geq y \geq \underline{x^2} \}$

$f(x,y) = 2 \cdot x \cdot y$



$$\int_A f \, d\mu = \int_{\mathbb{R}^2} f \cdot \chi_A \, d\mu = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y) \chi_A(x,y) \, dy \right) dx$$

$$= \int_0^1 \left(\int_{x^2}^{x} 2 \cdot x \cdot y \, dy \right) dx = 2 \int_0^1 x \left(\int_{x^2}^{x} y \, dy \right) dx$$

antiderivative: $\frac{1}{2} y^2$

$$= 2 \cdot \int_0^1 x \, \frac{1}{2} \left(x^2 - x^4 \right) dx = \int_0^1 (x^3 - x^5) \, dx = \frac{1}{12}$$