The Bright Side of Mathematics

Measure Theory – Part 6

$(X, \mathcal{A}, \mu)$ measure space

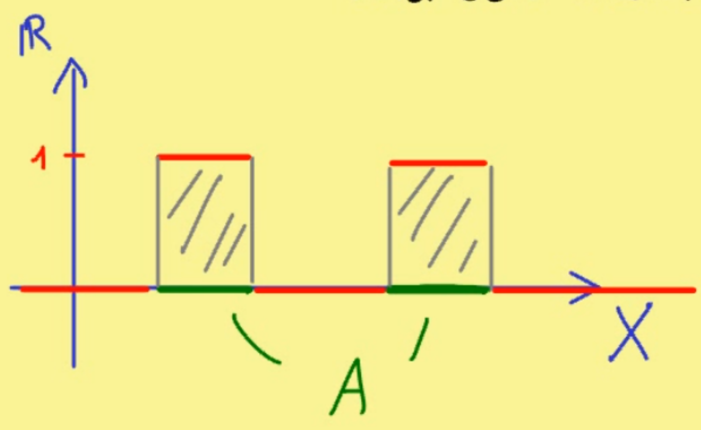
set

collection of subsets of X: σ-algebra

$\mu : \mathcal{A} \to [0, \infty]$ measure

Measurable maps $f: X \to \mathbb{R}$, $f^{-1}(E) \in \mathcal{A}$ for all Borel sets $E \subseteq \mathbb{R}$

σ-algebra $\mathcal{A}$ $\quad$ σ-algebra $\mathcal{B}$



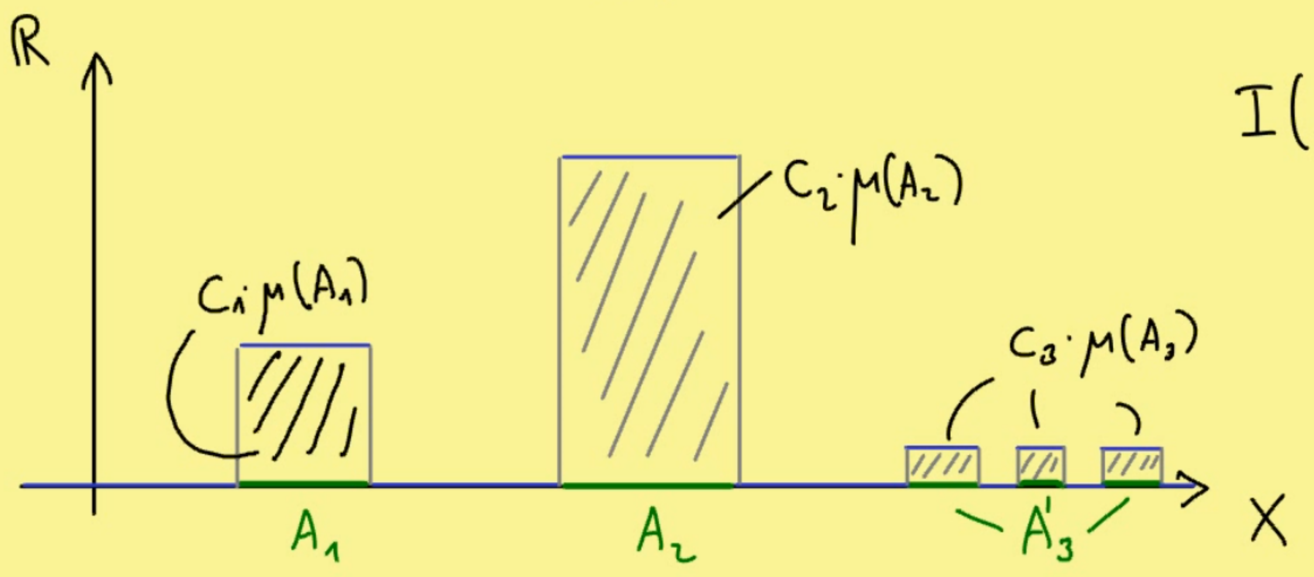
For example: $\chi_A : X \to \mathbb{R}$, $A \in \mathcal{A}$

$$I(\chi_A) := \mu(A)$$

Simple functions (step functions, staircase functions, ...)

For $A_1, \dots, A_n \in \mathcal{A}$, $c_1, \dots, c_n \in \mathbb{R}$

$$f(x) = \sum_{i=1}^{n} c_i \cdot \chi_{A_i}(x)$$ measurable



$$I(f) := \sum_{i=1}^{n} c_i \, \mu(A_i)$$

Integral

Problem: $3 \cdot \infty - 2 \cdot \infty$??

Definition: $\mathcal{S}^+ := \{ f: X \to \mathbb{R} \mid f \text{ simple function}, \ f \geq 0 \}$

· measurable

· only finitely many values

$f \in \mathcal{S}^+$ choose representation $f(x) = \sum_{i=1}^{n} c_i \, \chi_{A_i}(x)$, $c_i \geq 0$.

The (Lebesgue) integral f w.r.t. μ:

$$\int_X f(x) \cdot \; = \int_X f \, d\mu = I(f) = \sum_{i=1}^{n} c_i \cdot \mu(A_i) \in [0, \infty]$$

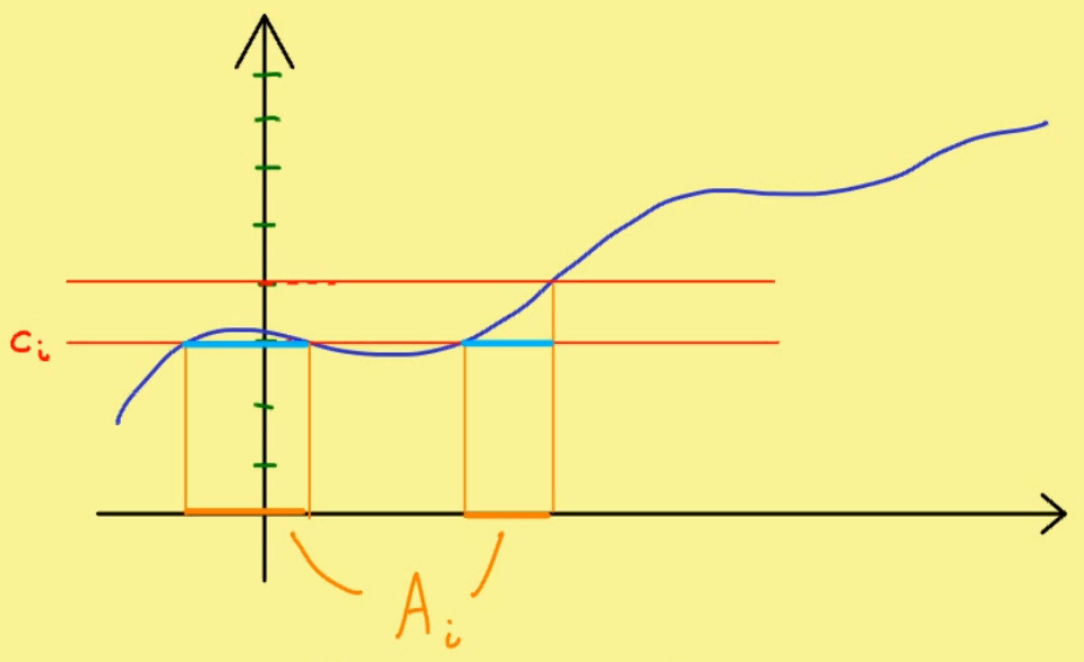
well-defined

Properties: (a) $I(\alpha f + \beta g) = \alpha \cdot I(f) + \beta \cdot I(g)$, $\alpha, \beta \geq 0$

(b) $f \leq g \Rightarrow I(f) \leq I(g)$ (monotonicity)

Definition:

$f: X \to [0, \infty)$ measurable



$$\int_X f \, d\mu := \sup \{ I(h) \mid h \in \mathcal{S}^+, \ h \leq f \} \qquad h = \sum_{i=1}^{n} c_i \cdot \chi_{A_i}$$

Lebesgue integral of f w.r.t. μ

f is call μ-integrable if $\int_X f \, d\mu < \infty$.