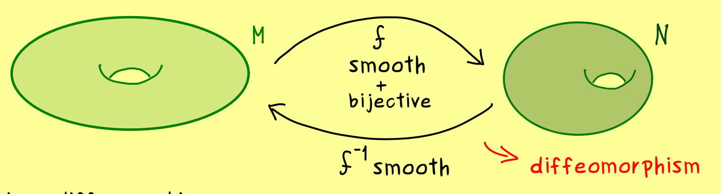
The Bright Side of Mathematics

Manifolds - Part 44

Change of variables:



If $f: M \longrightarrow N$ is a diffeomorphism and orientation preserving, then:

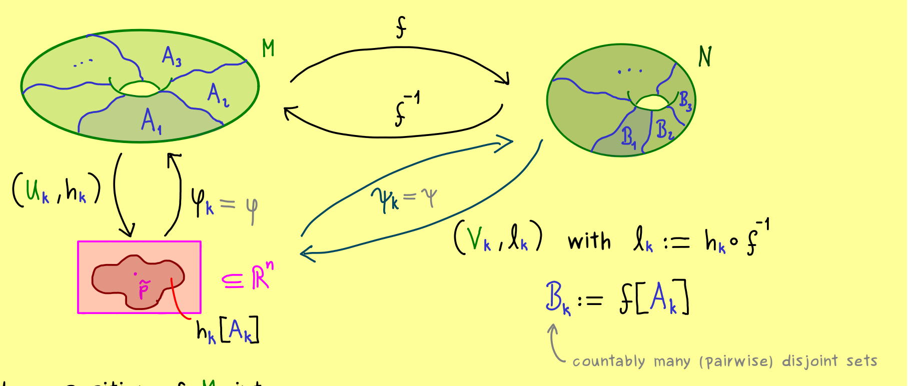
$$\int_M f^* \omega = \int_{f[M]} \omega$$

$(v_1, v_2, \dots, v_n)$ positively orientated in T_pM

$\Longrightarrow$ $(df_p(v_1), df_p(v_2), \dots, df_p(v_n))$ positively orientated in $T_{f(p)}N$

$$(f^*\omega)_p(v_1, v_2, \dots, v_n) = \omega_{f(p)}(df_p(v_1), df_p(v_2), \dots, df_p(v_n))$$

Proof:



decomposition of M into countably many (pairwise) disjoint sets $A_k \subseteq U_k$

$$\int_{A_k} f^*\omega = \int_{h_k[A_k]} \varphi^* f^* \omega \qquad \text{with} \quad \psi = f \circ \varphi$$

We have: $(\varphi^* f^* \omega)_x (u_1, u_2, \dots, u_n) = (f^*\omega)_{\varphi(x)} (d\varphi_x(u_1), \dots, d\varphi_x(u_n))$

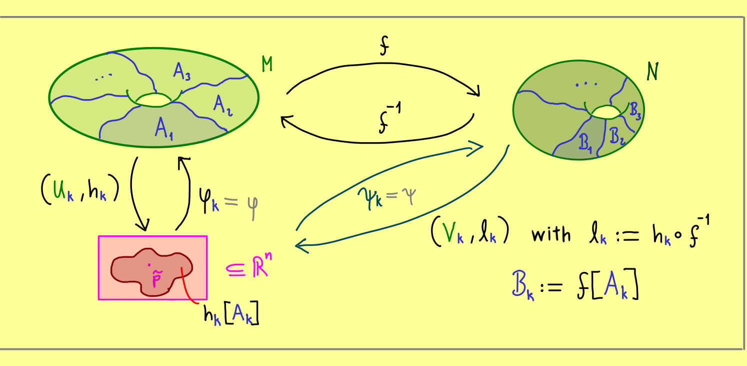
$$= \omega_{f(\varphi(x))} (df_{\varphi(x)} d\varphi_x(u_1), \dots, df_{\varphi(x)} d\varphi_x(u_n))$$

$$= \omega_{\psi(x)} (d\psi_x(u_1), \dots, d\psi_x(u_n))$$

$$= (\psi^* \omega)_x (u_1, u_2, \dots, u_n) \qquad \Longrightarrow (f \circ \varphi)^* = \varphi^* f^*$$

Result:

$$\int_{A_k} f^*\omega = \int_{h_k[A_k]} \varphi^* f^* \omega = \int_{h_k[A_k]} \psi^* \omega = \int_{\ell_k[B_k]} \psi^* \omega = \int_{B_k} \omega$$



$$\ell_k[B_k] = (h_k \circ f^{-1})[B_k] = h_k[A_k]$$

$\overset{\sum_k}{\Longrightarrow}$ (on both sides) $$\int_M f^*\omega = \int_{f[M]} \omega$$

□