BECOME A MEMBER

ON STEADY

The Bright Side of
Mathematics

Manifolds – Part 35

We already know: An orientable n-dimensional manifold M has a non-trivial volume form $\omega \in \Omega^n(M)$.

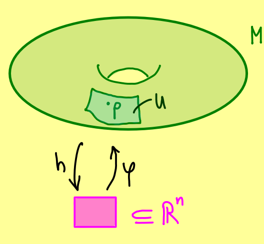
Definition: M orientable Riemannian manifold of dimension n.

Then the canonical volume form $\omega_M \in \Omega^n(M)$ is defined by:

If $(v_1, v_2, \dots, v_n)$ is a positively orientated basis of T_pM and an orthonormal basis of T_pM (ONB), [$g_p(v_i, v_j) = \delta_{ij}$]

then: $\omega_M(p)(v_1, v_2, \dots, v_n) = 1$

Proposition: (M, g) orientable Riemannian manifold of dimension n.



Let (U, h) be a chart such that the basis

$$\left(\partial_1^{(h)}(x), \partial_2^{(h)}(x), \dots, \partial_n^{(h)}(x)\right)$$

is positively orientated for all $x \in U$.

$$\omega_M(x) = \sqrt{\det(G)}\, dx_x^1 \wedge dx_x^2 \wedge \dots \wedge dx_x^n$$

(dual basis)

where $G_{ij} := g_x\left(\partial_i^{(h)}(x), \partial_j^{(h)}(x)\right)$

determinant of Gram / Gramian

Proof: $\left(\partial_1^{(h)}(x), \partial_2^{(h)}(x), \dots, \partial_n^{(h)}(x)\right) \xrightarrow{\text{Gram–Schmidt}} (v_1, v_2, \dots, v_n)$ ONB

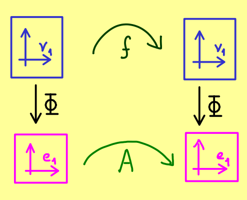
positively orientated; f linear map (back arrow); positively orientated

Then: $\omega_M(x)\left(\partial_1^{(h)}(x), \partial_2^{(h)}(x), \dots, \partial_n^{(h)}(x)\right)$

$$= \omega_M(x)\big(f(v_1), f(v_2), \dots, f(v_n)\big) = f^*\omega_M(x)(v_1, \dots, v_n)$$

$$= \det(f)\, \underbrace{\omega_M(x)(v_1, \dots, v_n)}_{=1}$$

$$g_x\left(\partial_i^{(h)}(x), \partial_j^{(h)}(x)\right) = g_x\big(f(v_i), f(v_j)\big)$$



$$= g_x\left(\Phi^{-1} A \Phi(v_i), \Phi^{-1} A \Phi(v_j)\right)$$

$$= \langle A \underbrace{\Phi(v_i)}_{e_i}, A \underbrace{\Phi(v_j)}_{e_j} \rangle_{\text{standard}} = (A^T A)_{ij}$$

$$\Rightarrow \det(G) = \det(A)^2$$

□