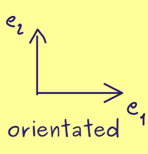
The Bright Side of Mathematics

Manifolds – Part 31

vector space ⟵ orientation

for example: $\mathbb{R}^n$ with basis: $\mathcal{B} = (e_1, e_2, \dots, e_n)$ ← positively orientated



change-of-basis matrix → $T_{\mathcal{C} \leftarrow \mathcal{B}}$

$\mathcal{C} = (c_1, c_2, \dots, c_n)$

two cases:

$\det(T_{\mathcal{C} \leftarrow \mathcal{B}}) > 0$: positively orientated

$\det(T_{\mathcal{C} \leftarrow \mathcal{B}}) < 0$: negatively orientated

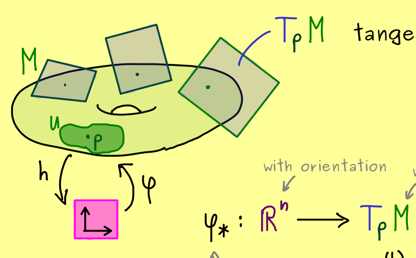
$\Longrightarrow$ two equivalence classes for bases

$\big[\{\text{basis}\} \big]$ with $\det(T_{\mathcal{C} \leftarrow \mathcal{B}}) > 0$ $\qquad$ $\big[\{\text{basis}\} \big]$ with $\det(T_{\mathcal{C} \leftarrow \mathcal{B}}) > 0$

Remember: V finite-dimensional vector space + one chosen equivalence class

⇝ orientation (V, or)

Orientations for manifolds:



$\varphi_* : \mathbb{R}^n \longrightarrow T_pM$ (both with orientation)

$e_j \longmapsto \partial_j^{(h)}(p)$

should not change orientation

Definition: A smooth manifold M is called orientable if there is a family of orientations for the tangent spaces $\{(T_pM, or_p)\}$ such that

$$\forall p \in M \;\; \exists (U, h) \;\; \forall x \in U : \quad \left(\partial_1^{(h)}(x), \partial_2^{(h)}(x), \dots, \partial_n^{(h)}(x)\right) \in or_x$$

Example: (a) If M has an atlas with one chart (M, h), then M is orientable.

(b) Möbius strip: