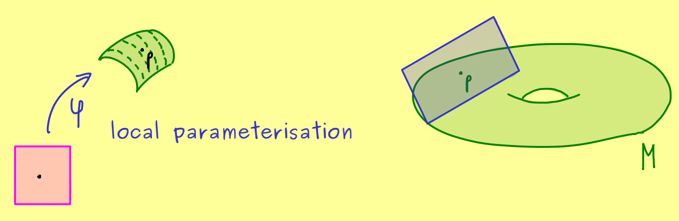
The Bright Side of Mathematics

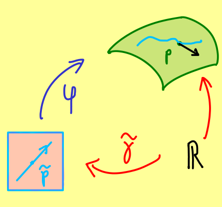
Manifolds – Part 20

$T_p^{sub} M$ tangent space for submanifold $M \subseteq \mathbb{R}^n$, $p \in M$



$$T_p^{sub} M := \left\{ J_\varphi(\varphi^{-1}(p))\, x \;\middle|\; x \in \mathbb{R}^k \right\} \subseteq \mathbb{R}^n$$

Idea:



parameterised curve $\gamma : \mathbb{R} \to M$

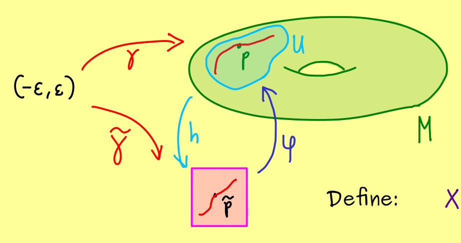
Proposition: $T_p^{sub} M = \left\{ \gamma'(0) \;\middle|\; \gamma : (-\varepsilon, \varepsilon) \to M \text{ differentiable with } \gamma(0) = p \right\}$

Proof: $(\subseteq)$ $v \in T_p^{sub} M \Rightarrow v = J_\varphi(\underbrace{\varphi^{-1}(p)}_{\tilde{p}})\, x$ for $x \in \mathbb{R}^k$, φ local parameterisation

$$\Rightarrow v = J_\varphi(\tilde{\gamma}(0))\, \tilde{\gamma}'(0) \quad \text{with } \tilde{\gamma}(t) = \tilde{p} + t x, \; \tilde{\gamma} : (-\varepsilon, \varepsilon) \to \mathbb{R}^k$$

$$= \frac{d}{dt} (\underbrace{\varphi \circ \tilde{\gamma}}_{\gamma}) \Big|_{t=0} = \gamma'(0)$$

$(\supseteq)$ Take: $\gamma : (-\varepsilon, \varepsilon) \to M$ differentiable with $\gamma(0) = p$



Define: $x := \frac{d}{dt} (\underbrace{h \circ \gamma}_{\tilde{\gamma}}) \Big|_{t=0}$

$$\gamma'(0) = \frac{d}{dt} (\varphi \circ \tilde{\gamma}) \Big|_{t=0} = J_\varphi(\tilde{\gamma}(0))\, \tilde{\gamma}'(0) = J_\varphi(\varphi^{-1}(p))\, x \in T_p^{sub} M$$