The Bright Side of Mathematics

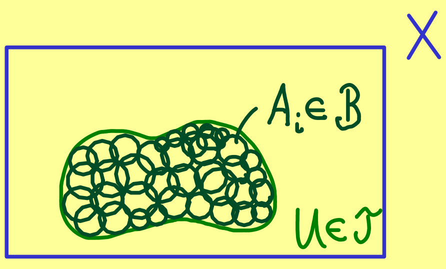
Manifolds – Part 6

$(X, \mathcal{T})$ topological space: generate the topology $\mathcal{T}$

Definition: Let $(X, \mathcal{T})$ be a topological space. A collection of open subsets $\mathcal{B} \subseteq \mathcal{T}$ is called a basis (base) of $\mathcal{T}$ if:

for all $U \in \mathcal{T}$ there is $(A_i)_{i \in I}$ with $A_i \in \mathcal{B}$

and $\bigcup_{i \in I} A_i = U$



Examples: (a) $\mathcal{B} = \mathcal{T}$ is always a basis.

(b) If $\mathcal{T}$ is discrete topology on X, then $\mathcal{B} = \{ \{x\} \mid x \in X \}$ is a basis of $\mathcal{T}$.

(c) Let $(X, \mathcal{T})$ be the topological space induced by a metric space (X, d)

$$\mathcal{B} = \{ B_\varepsilon(x) \mid x \in X, \varepsilon > 0 \}$$ is a basis of $\mathcal{T}$.

(d) $\mathbb{R}^n$ with standard topology (defined by Euclidean metric)

$$\mathcal{B} = \{ B_\varepsilon(x) \mid x \in \mathbb{Q}^n, \varepsilon \in \mathbb{Q}, \varepsilon > 0 \}$$ is a basis of $\mathcal{T}$.

only countably many elements

Definition: A topological space $(X, \mathcal{T})$ is called second-countable if there is a countable basis of $\mathcal{T}$.