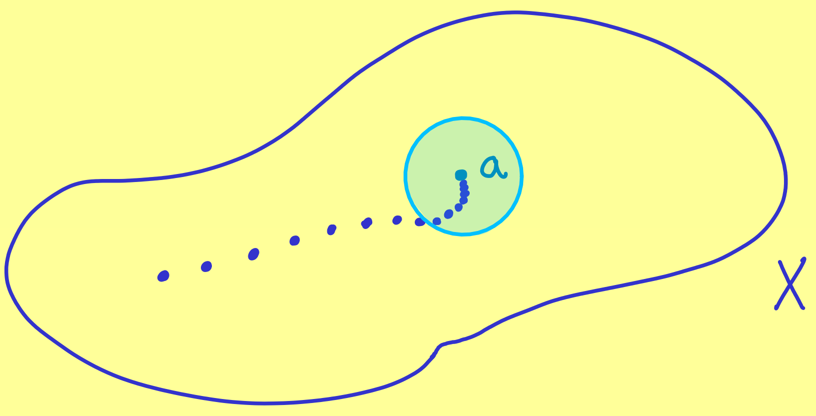
# Manifolds – Part 3

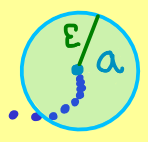
$(X, \mathcal{T})$ topological space




**Convergence:** $(a_n)_{n \in \mathbb{N}}$ , $a_n \in X$

converges to $a \in X$

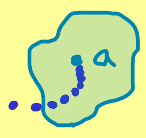
**In a metric space:**




The sequence members lie in each $\varepsilon$-ball around $a$, eventually.

For each $\varepsilon$-ball $B_\varepsilon(a)$, there is $N \in \mathbb{N}$ such that for all $n \geq N$ : $a_n \in B_\varepsilon(a)$

**In a topological space:**




The sequence members lie in each $\underbrace{\text{open neighbourhood of } a}_{\text{an open set } U \in \mathcal{T} \text{ with } a \in U}$, eventually.

**Definition:** $(X, \mathcal{T})$ topological space, $(a_n)_{n \in \mathbb{N}}$ sequence in $X$.

$a_n \xrightarrow{n \to \infty} a$ $:\Longleftrightarrow$ For each $U \in \mathcal{T}$ with $a \in U$, there is $N \in \mathbb{N}$ such that for all $n \geq N$ : $a_n \in U$

**Example:** $X = \mathbb{R}$, $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(b, \infty) \mid b \in \mathbb{R}\}$

$$(a_n)_{n \in \mathbb{N}} = \left(\frac{1}{n}\right)_{n \in \mathbb{N}}$$

- converges to $0$: each open neighbourhood of $0$ looks like $(b, \infty)$ for $b < 0$, so: $\frac{1}{n} \in (b, \infty)$
- converges to $-1$: each open neighbourhood of $-1$ looks like $(b, \infty)$ for $b < -1$, so: $\frac{1}{n} \in (b, \infty)$
- converges to $-2$

**Definition:** A topological space $(X, \mathcal{T})$ is called a Hausdorff space if for all $x, y \in X$ with $x \neq y$ there is an open neighbourhood of $x$: $U_x \in \mathcal{T}$ and there is an open neighbourhood of $y$: $U_y \in \mathcal{T}$ with: $U_x \cap U_y = \emptyset$