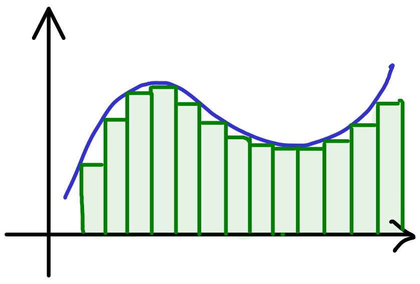
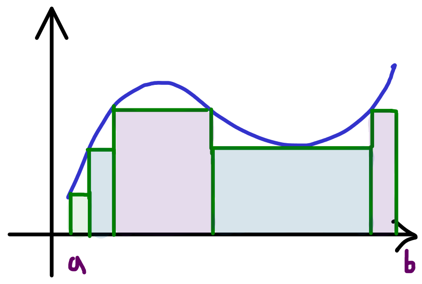
Manifolds – Part 38

Integration: $f: \mathbb{R} \to \mathbb{R}$ (smooth function later)



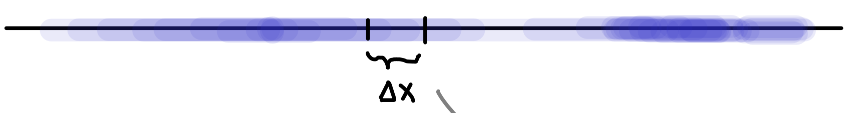
Riemann integral

$$\int_{[a,b]} f(x)\, dx$$



Lebesgue integral

See $f(x)$ as a density at point $x \in \mathbb{R}$:

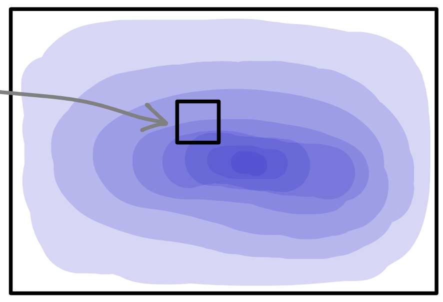


$f(x) \cdot \Delta x$

density · length = mass

$$\sum f(x) \cdot \Delta x \rightsquigarrow \int_{\mathbb{R}} f(x)\, dx = \text{total mass}$$

Same idea in higher dimensions: $f: \mathbb{R}^2 \to \mathbb{R}$



density · area = mass

$$\rightsquigarrow \int_{\mathbb{R}^2} f(x,y)\, d(x,y) = \text{total mass}$$

Let's take $M = \mathbb{R}^2$: differential form $\omega : p \mapsto f(p)\, dx \wedge dy \in \mathrm{Alt}^2(\underbrace{T_p M}_{= \mathbb{R}^2})$

$$\rightsquigarrow \quad \omega_p(v, w) = f(p) \Big(\underbrace{dx(v)}_{v_1} \cdot \underbrace{dy(w)}_{w_2} - \underbrace{dx(w)}_{w_1} \cdot \underbrace{dy(v)}_{v_2} \Big)$$

$$= f(p) \det(v, w)$$

integral: $\displaystyle \int_M \omega := \int_M f \, dx \wedge dy = \int_{\mathbb{R}^2} f(x,y) \, d(x,y)$