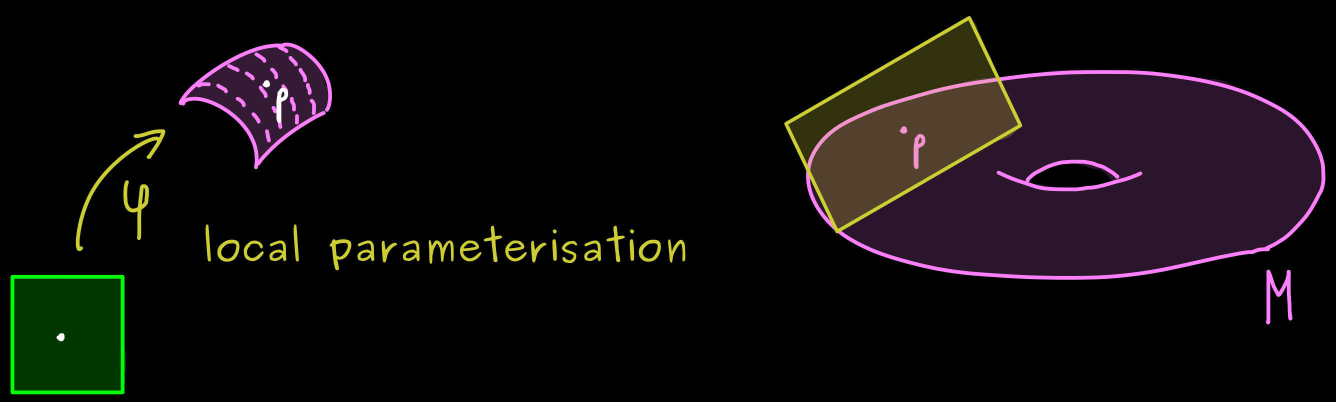
The Bright Side of Mathematics

The following pages cover the whole Manifolds course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

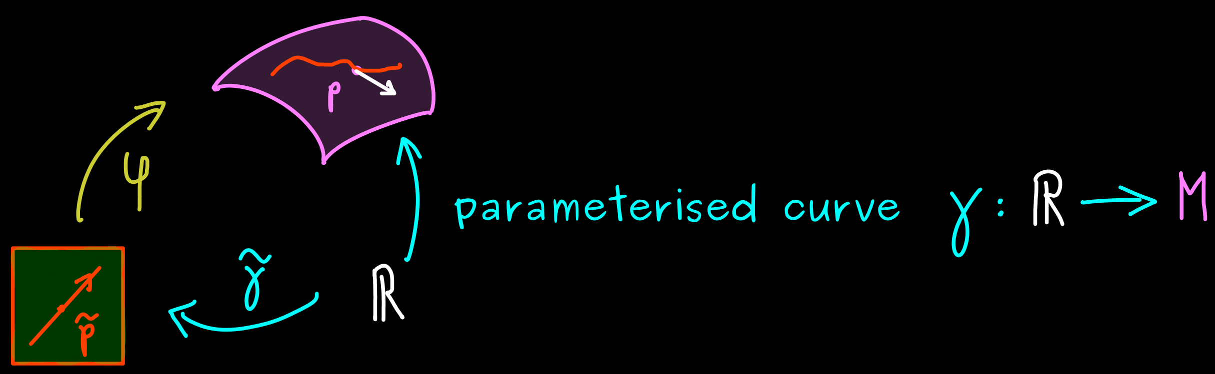
Manifolds – Part 20

$T_p^{\text{sub}} M$ tangent space for submanifold $M \subseteq \mathbb{R}^n$, $p \in M$



$$T_p^{\text{sub}} M := \left\{ J_\varphi(\varphi^{-1}(p))\, x \;\middle|\; x \in \mathbb{R}^k \right\} \subseteq \mathbb{R}^n$$

Idea:



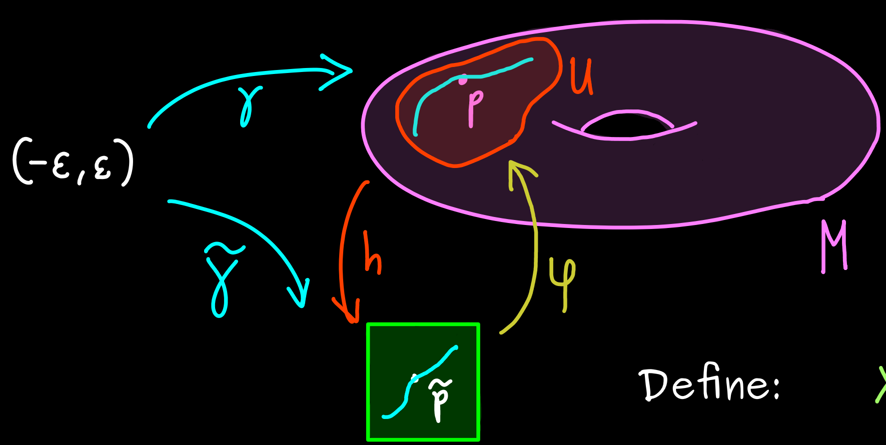
Proposition: $$T_p^{\text{sub}} M = \left\{ \gamma'(0) \;\middle|\; \gamma : (-\varepsilon, \varepsilon) \to M \text{ differentiable with } \gamma(0) = p \right\}$$

Proof: $(\subseteq)$ $v \in T_p^{\text{sub}} M \Rightarrow v = J_\varphi(\underbrace{\varphi^{-1}(p)}_{\tilde{p}})\, x$ for $x \in \mathbb{R}^k$, φ local parameterisation

$$\Rightarrow v = J_\varphi(\tilde{\gamma}(0))\, \tilde{\gamma}'(0) \qquad \text{with } \tilde{\gamma}(t) = \tilde{p} + t x, \; \tilde{\gamma} : (-\varepsilon, \varepsilon) \to \mathbb{R}^k$$

$$= \frac{d}{dt} (\underbrace{\varphi \circ \tilde{\gamma}}_{\gamma}) \Big|_{t=0} = \gamma'(0)$$

$(\supseteq)$ Take: $\gamma : (-\varepsilon, \varepsilon) \to M$ differentiable with $\gamma(0) = p$



Define: $X := \frac{d}{dt} (\underbrace{h \circ \gamma}_{\tilde{\gamma}}) \Big|_{t=0}$

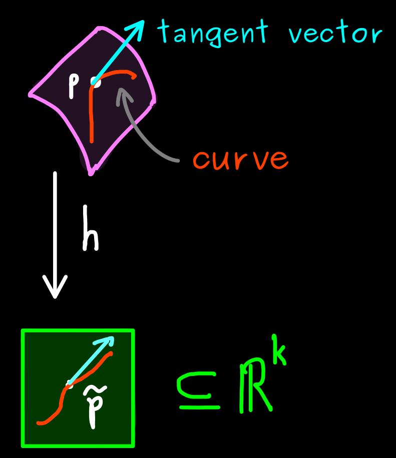
$$\gamma'(0) = \frac{d}{dt} (\varphi \circ \tilde{\gamma}) \Big|_{t=0} = J_\varphi(\tilde{\gamma}(0))\, \tilde{\gamma}'(0) = J_\varphi(\varphi^{-1}(p))\, x \in T_p^{\text{sub}} M$$

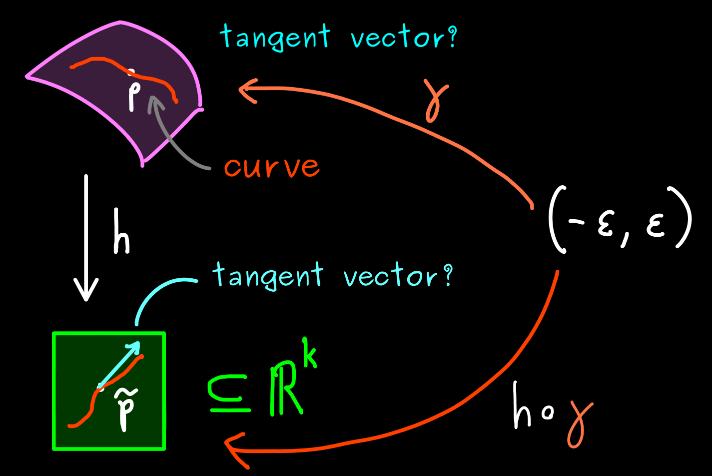
Manifolds - Part 21

$$T_p^{sub} M \rightsquigarrow T_p M$$

for $M \subseteq \mathbb{R}^n$ smooth submanifold

for M smooth manifold





Definition: $C_p(M) := \{ \gamma : (-\varepsilon, \varepsilon) \to M \mid \gamma \text{ differentiable with } \gamma(0) = p \}$

$$\gamma \sim \alpha \;:\Longleftrightarrow\; (h \circ \gamma)'(0) = (h \circ \alpha)'(0)$$

for a chart (U, h).

equivalent class: $[\gamma]_\sim := \{ \alpha \mid \gamma \sim \alpha \}$ represents tangent vector

$$T_p M := C_p(M) / \sim \qquad \text{(set of all equivalence classes)}$$

tangent space of the manifold M

Result:

- For a submanifold $T_p^{sub} M \xleftrightarrow[\text{bijection}]{} T_p M$

$$\gamma'(0) \longleftrightarrow [\gamma]_\sim$$

- $T_p M$ is a vector space with the operations:

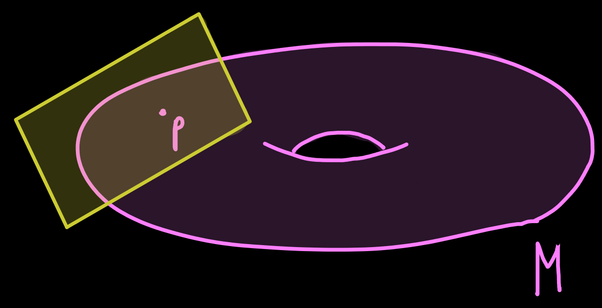
$$v + w := h_*^{-1}\big(h_*(v) + h_*(w)\big) \qquad \text{with } h_* : [\gamma]_\sim \mapsto \underbrace{(h \circ \gamma)'(0)}_{\in \mathbb{R}^k}$$

$$\lambda \cdot v := h_*^{-1}\big(\lambda \cdot h_*(v)\big)$$

Manifolds – Part 22

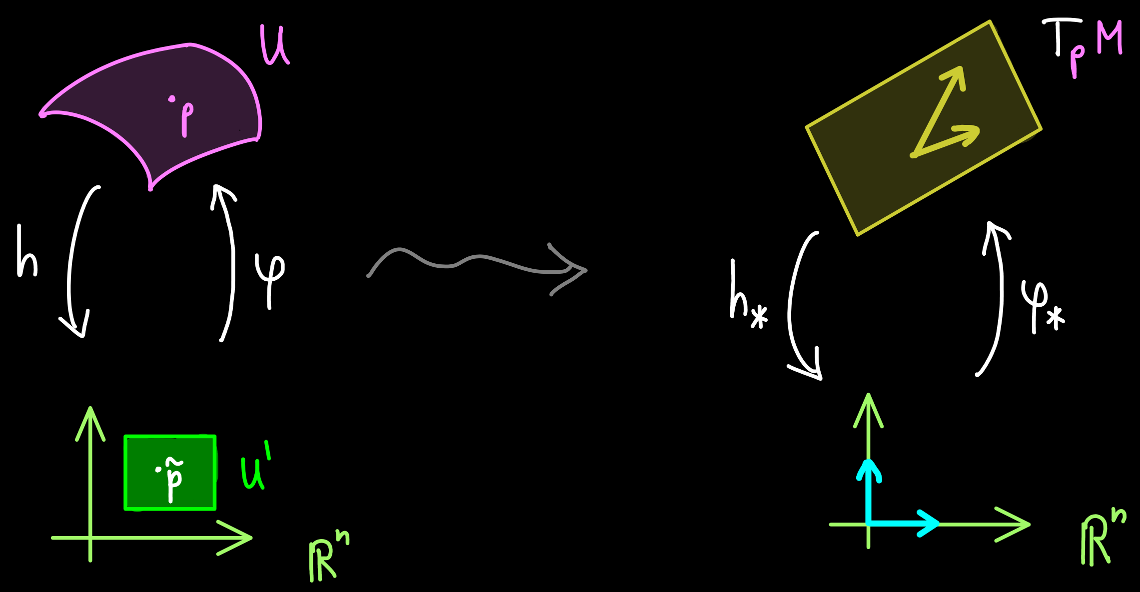
smooth manifold M of dimension n, $p \in M$.

$T_p M$



well–defined and with dimension n

chart (U,h):



defined by:

$h_* : T_p M \longrightarrow \mathbb{R}^n$

$[\gamma] \mapsto (h \circ \gamma)'(0)$

linear + bijective

$\varphi_* := h_*^{-1}$

<u>Definition:</u> <u>coordinate basis</u> (standard basis with respect to (U,h)):

For (U,h) and $p \in U$, we define: $\quad \partial_j := \varphi_*(e_j)$

where $(e_1, e_2, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$

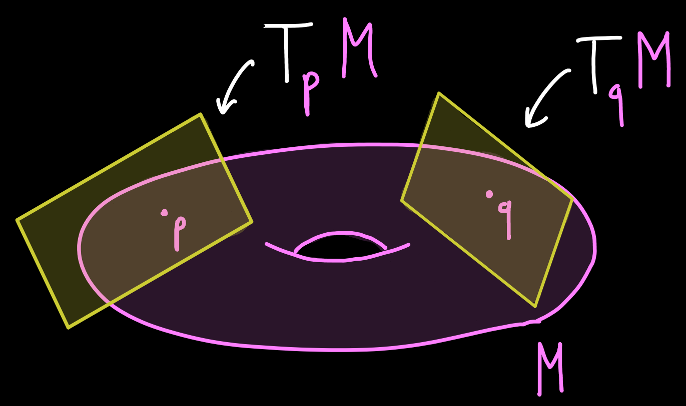
<u>Remember:</u> For submanifolds:

$$\begin{tikzcd}
T_p M \arrow[r] & T_p^{\text{sub}} M \\
\mathbb{R}^n \arrow[u, "\varphi_*"] \arrow[ur, "J_\varphi(\tilde{p})"'] &
\end{tikzcd}$$

$(\partial_1, \partial_2, \dots, \partial_n)$ is essentially $\left(\frac{\partial \varphi}{\partial x_1}(\tilde{p}), \frac{\partial \varphi}{\partial x_2}(\tilde{p}), \dots, \frac{\partial \varphi}{\partial x_n}(\tilde{p}) \right)$

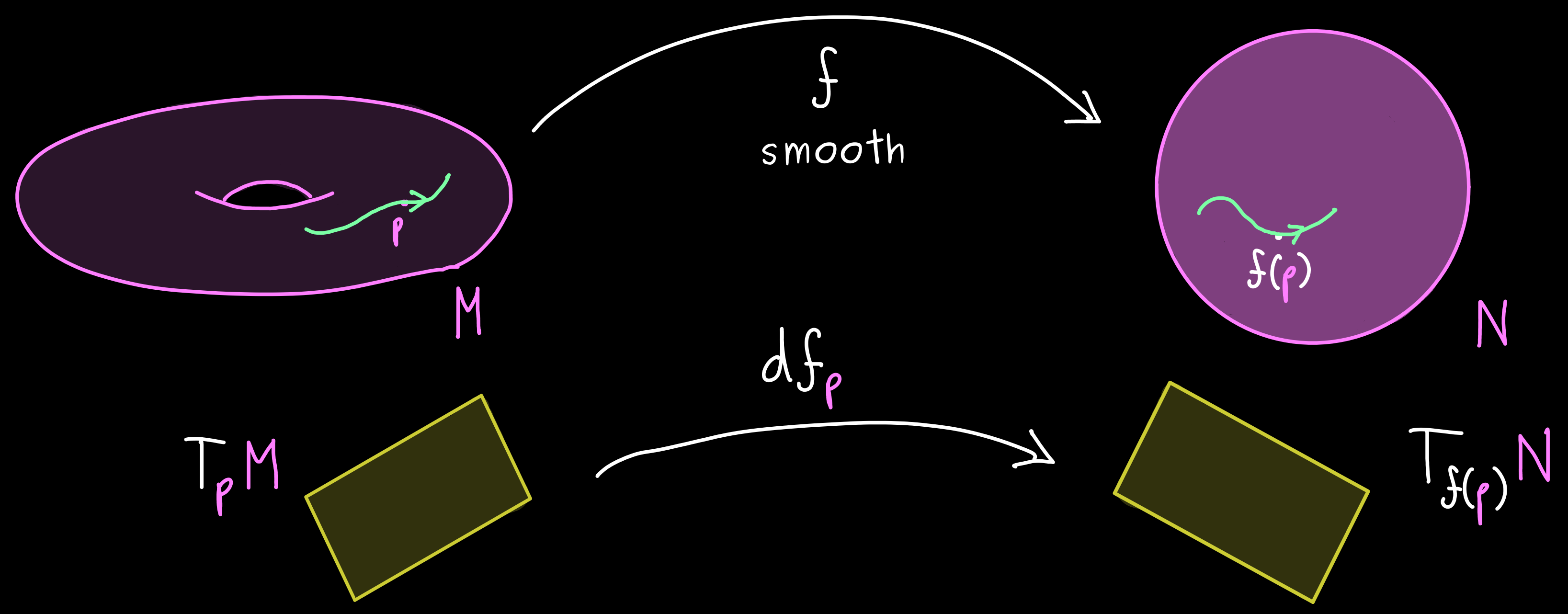
<u>Soon:</u> $f : M \longrightarrow N$ smooth $\rightsquigarrow$ $df_p : T_p M \longrightarrow T_p N$ differential

Manifolds – Part 23



Definition: tangent bundle $TM := \underset{p\in M}{\overset{\text{disjoint}}{\bigsqcup}} T_pM := \bigcup_{p\in M} \{p\} \times T_pM$

$\hookrightarrow$ smooth manifold of dimension $2\cdot \dim(M)$



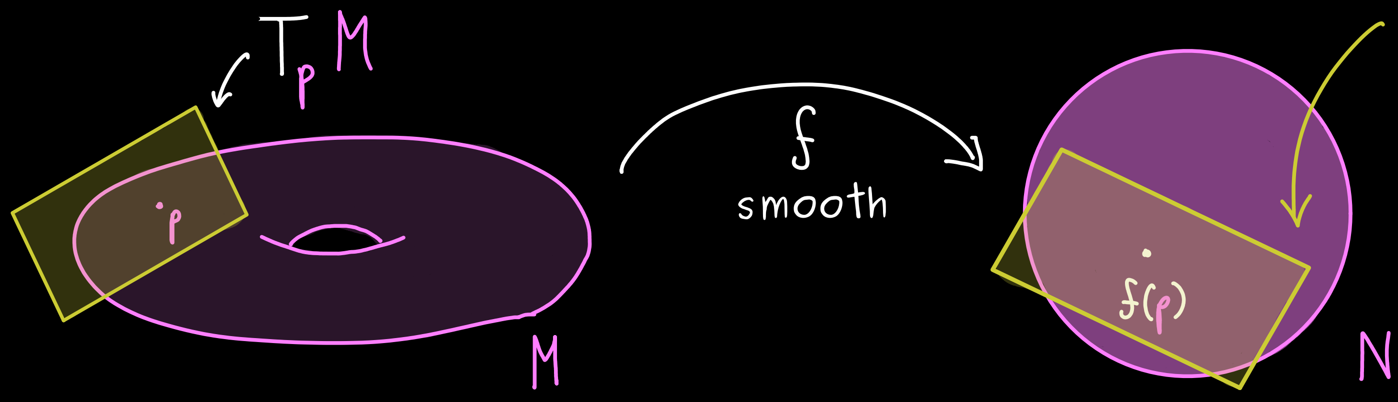
Definition: differential of f at point p

$$df_p : T_pM \longrightarrow T_{f(p)}N$$
$$[\gamma] \longmapsto [f\circ\gamma]$$

differential: $df : p \longmapsto df_p$

Example for submanifolds $\quad M, N \subseteq \mathbb{R}^n$ smooth submanifolds

$$T_{f(p)}N \overset{\text{bijection}}{=} T^{\text{sub}}_{f(p)}N$$



$$[\gamma] \xmapsto{\;df_p\;} [f\circ\gamma] \overset{\text{bijection}}{=} (f\circ\gamma)'(0)$$

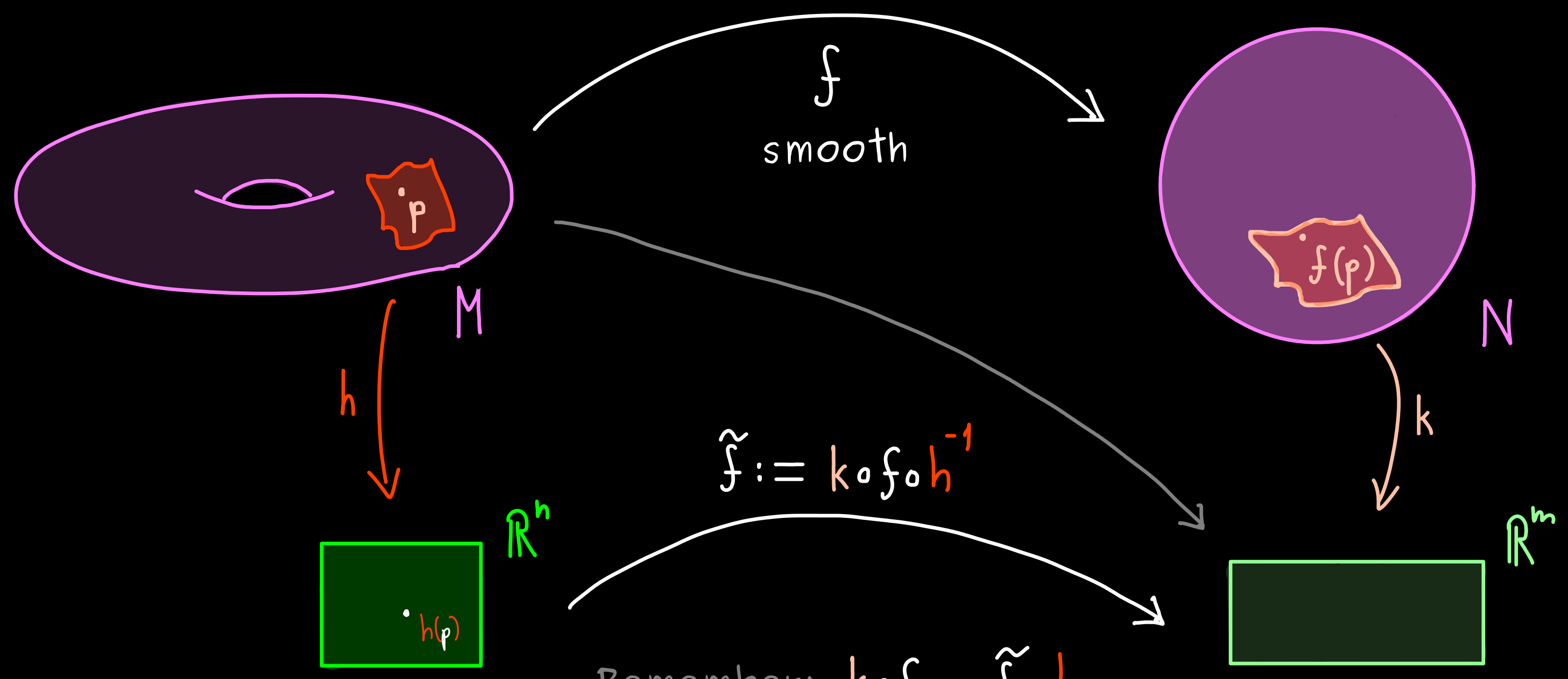
Example: $\quad f: \mathbb{R}^n \to \mathbb{R}$ (smooth map)

$$df_p([\gamma]) \overset{\text{bijection}}{=} (f\circ\gamma)'(0) = J_f(\underbrace{\gamma(0)}_{p})\,\underbrace{\gamma'(0)}_{\text{tangent vector}}$$

$=$ directional derivative of f along $[\gamma]$ at p

Manifolds – Part 24

Differential in local charts?



Remember: $k \circ f = \tilde{f} \circ h$

Choose: $[\gamma] \in T_pM$:

$$dk_{f(p)}\big(df_p([\gamma])\big) = dk_{f(p)}\big([f \circ \gamma]\big)$$

$$= [k \circ f \circ \gamma] \overset{\text{bijection}}{=} (k \circ f \circ \gamma)'(0)$$

$$= (\tilde{f} \circ h \circ \gamma)'(0)$$

ordinary chain rule

$$= J_{\tilde{f}}(h(p))\,(h \circ \gamma)'(0)$$

$$\overset{\text{bijection}}{=} J_{\tilde{f}}(h(p))\,[h \circ \gamma]$$

$$= J_{\tilde{f}}(h(p))\,dh_p([\gamma])$$

Remember:

$$\boxed{\begin{aligned} f &= k^{-1} \circ \tilde{f} \circ h \\ df &= dk^{-1}\, J_{\tilde{f}}\, dh \end{aligned}}$$

Manifolds – Part 25

Recall: $p \in M$, (U,h) : coordinate basis $(\partial_1, \dots, \partial_n)$ of T_pM

$$\varphi = h^{-1}, \quad \partial_j := \varphi_*(e_j) = d\varphi_{h(p)}(e_j)$$

defined by:
$h_* : T_pM \to \mathbb{R}^n$
$[\gamma] \mapsto (h\circ\gamma)'(0)$
linear + bijective
$\varphi_* := h_*^{-1}$

Directional derivative: $f : M \to \mathbb{R}$ smooth

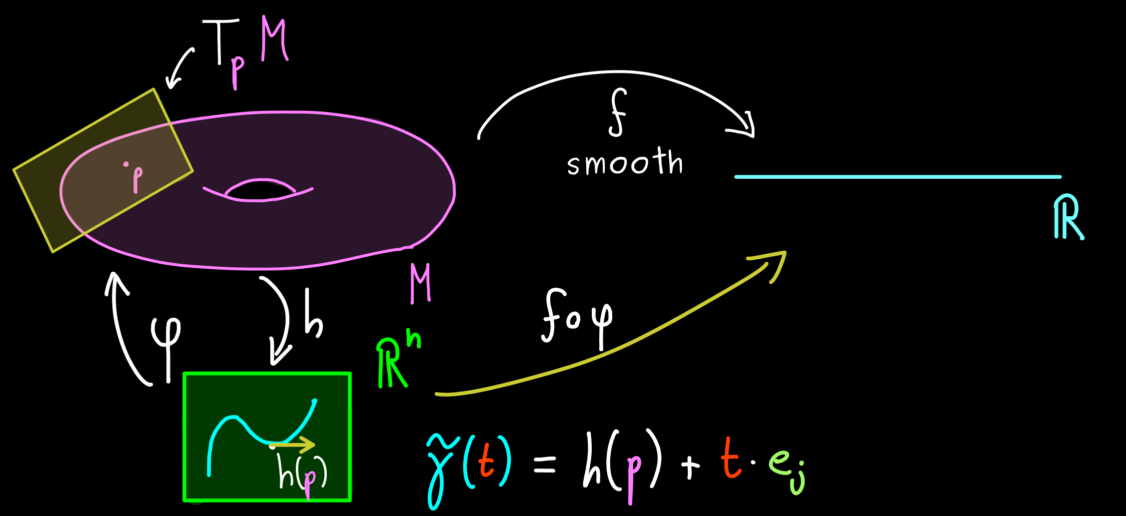
$$(\partial_j f)(p) := df_p(\partial_j)$$

$$= df_p(d\varphi_{h(p)}(e_j))$$

$$= [f\circ\varphi\circ\tilde\gamma]$$

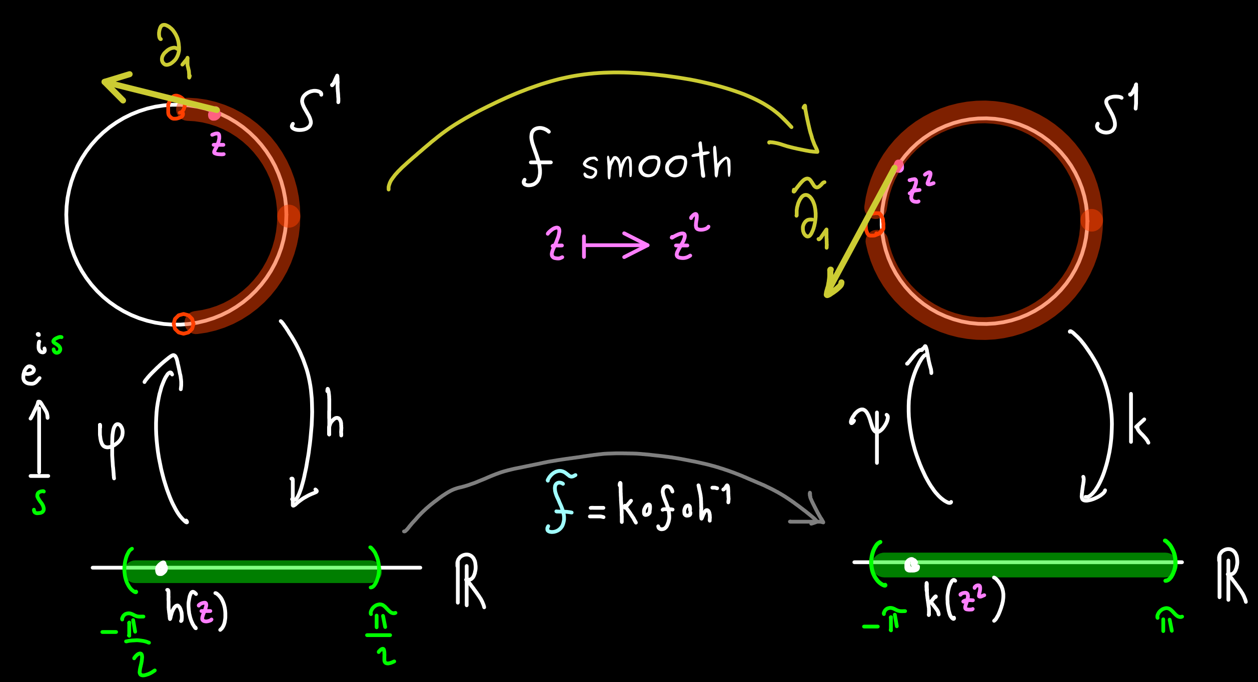
$$\overset{\text{bijection}}{=} (f\circ\varphi\circ\tilde\gamma)'(0)$$

$$\overset{\text{chain rule}}{=} J_{f\circ\varphi}(h(p))\, \underbrace{\tilde\gamma'(0)}_{e_j} = \frac{\partial(f\circ\varphi)}{\partial x_j}(h(p))$$



$$\tilde\gamma(t) = h(p) + t\cdot e_j$$

Example:



$$\partial_1 = d\varphi_{h(z)}(e_1) = [\varphi\circ\tilde\gamma], \quad \tilde\gamma(t) = \underbrace{h(z)}_{s} + t$$

$$= (\varphi\circ\tilde\gamma)'(0) = \frac{d}{dt}\Big|_{t=0} e^{i(s+t)} = i\cdot e^{is}$$

$$\tilde\partial_1 = d\psi_{k(f(z))}(e_1)$$

$$= (\psi\circ\tilde\gamma)'(0) \qquad \tilde\gamma(t) = \underbrace{k(z^2)}_{2s} + t$$

$$= i\cdot e^{2is} \qquad 2s \mapsto (e^{is})^2$$

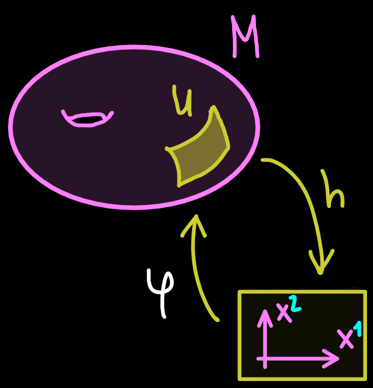
map $\tilde f$: $s \overset{\varphi}{\mapsto} e^{is} \overset{f}{\mapsto} (e^{is})^2 \overset{k}{\mapsto} 2s$

$$J_{\tilde f}(s) = 2$$

differential of f:

$$df_z(\partial_1) \overset{\text{last video}}{=} dk^{-1}_{z^2}\, \underbrace{J_{\tilde f}(h(p))}_{2}\, \underbrace{dh_z(\partial_1)}_{e_1} = 2\cdot dk^{-1}_{z^2}(e_1) = 2\cdot\tilde\partial_1$$

Manifolds – Part 26



Introduction to Ricci calculus / tensor calculus

↳ calculating in coordinates

↳ positions of indices matter (superscripts, subscripts)

our language	Ricci calculus
components of a given chart (U,h), $h: U \to \mathbb{R}^n$	$h^j : U \to \mathbb{R}$ coordinates or simply: $x^1, x^2, \dots, x^n$
coordinate basis of T_pM: $\partial_j := \varphi_*(e_j)$	$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}$
tangent vector $[\gamma] \in T_pM$: $v_1\partial_1 + v_2\partial_2 + \cdots + v_n\partial_n$	$v^1 \frac{\partial}{\partial x^1} + \cdots + v^n \frac{\partial}{\partial x^n} =: v^j \frac{\partial}{\partial x^j}$ (Einstein summation convention) contravariant vector
Later: inner product on T_pM: $\langle v, w \rangle \in \mathbb{R}$	$v^j \underbrace{g_{jk}}_{\text{tensor}} w^k$

dual to a contravariant vector: $v_j \underbrace{dx^j}$

↳ one-form (⇝linear map)

$$dx_j(\partial_k) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

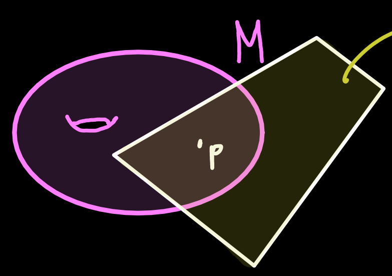
$$= \delta_{jk}$$

Kronecker delta

$$dx^j\left(\frac{\partial}{\partial x^k}\right) = \delta^j{}_k$$

Manifolds – Part 27

Recall:



T_pM n-dimensional vector space

Define: $T_p^*M := (T_pM)^*$

$$= \{ \alpha : T_pM \to \mathbb{R} \text{ linear} \}$$

$$\rightsquigarrow \quad dx_{j,p} : T_pM \to \mathbb{R}$$

$$dx_{j,p}(\partial_k) = \delta_{jk} \qquad \text{linear map!}$$

differential form: (one-form) map ω defined on M such that $\omega(p) \in T_p^*M$

$$dx_j : \; p \mapsto dx_{j,p} \in T_p^*M$$

Some multilinear algebra: $\mathrm{Alt}^k(V) := \Big\{ \alpha : \underbrace{V \times \cdots \times V}_{k\text{-times}} \to \mathbb{R} \;\; \text{multilinear } (k\text{-linear}) + \text{alternating} \Big\}$

alternating: $\alpha(v_1, \dots, v_k) = 0$ if $(v_1, \dots, v_k)$ linearly dependent

Example: $\alpha \in \mathrm{Alt}^2(V)$, $\alpha(v_1, v_2) = -\alpha(v_2, v_1)$

$$\det \in \mathrm{Alt}^2(\mathbb{R}^2)$$

$\alpha \in \mathrm{Alt}^k(V)$ is called an alternating k-form on V

Remember: $\mathrm{Alt}^1(V) = V^*$ (dual space of V)

$$\mathrm{Alt}^0(V) = \mathbb{R}$$

Manifolds – Part 28

Wedge product: $\wedge$ multiplication defined for $\alpha \in \mathrm{Alt}^k(V)$, $\beta \in \mathrm{Alt}^s(V)$

$$\wedge : \mathrm{Alt}^k(V) \times \mathrm{Alt}^s(V) \longrightarrow \mathrm{Alt}^{k+s}(V)$$
$$(\alpha, \beta) \longmapsto \alpha \wedge \beta$$

(k+s)-linear

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+s}) :\neq \alpha(v_1, \dots, v_k) \cdot \beta(v_{k+1}, \dots, v_{k+s})$$

not a possible definition!
(not alternating)

Definition: For $\alpha \in \mathrm{Alt}^k(V)$, $\beta \in \mathrm{Alt}^s(V)$, we define $\alpha \wedge \beta \in \mathrm{Alt}^{k+s}(V)$ by:

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+s}) := \frac{1}{k! \cdot s!} \sum_{\sigma \in S_{k+s}} \mathrm{sgn}(\sigma)\, \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+s)})$$

Examples: (a) $\alpha, \beta \in \mathrm{Alt}^1(V) = V^*$:

$$(\alpha \wedge \beta)(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

(b) $\alpha, \beta \in \mathrm{Alt}^1(\mathbb{R}^3)$, $\alpha\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = x_1$, $\beta\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = x_2 = \underbrace{(0, 1, 0)}_{\text{identified with } \beta} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$(\alpha \wedge \beta)\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = x_1 y_2 - y_1 x_2 = \left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{identified with } \alpha \wedge \beta} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle$$

Properties:

(a) $\alpha \wedge \beta = (-1)^{k \cdot s} \beta \wedge \alpha$ (anticommutative)

(b) $(\alpha + \alpha') \wedge \beta = \alpha \wedge \beta + \alpha' \wedge \beta$

$(\lambda \alpha) \wedge \beta = \lambda (\alpha \wedge \beta)$ (bilinear)

(c) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$ (associative)

(d) For a linear map $f: W \to V$ and $\alpha \in \mathrm{Alt}^k(V)$ define:

pullback $(f^*\alpha)(w_1, \dots, w_k) := \alpha(f(w_1), \dots, f(w_k))$

$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$ ("natural")

Manifolds – Part 29

M smooth manifold of dimension n $\Longrightarrow$ T_pM n-dimensional vector space

Definition:

$$\omega : M \longrightarrow \bigcup_{p \in M} \mathrm{Alt}^k(T_pM)$$

$$p \longmapsto \omega_p = \omega(p) \in \mathrm{Alt}^k(T_pM)$$

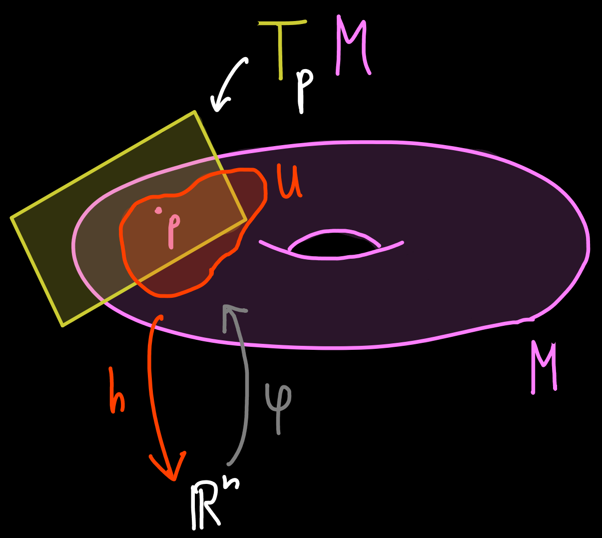
is called a k-form on M.

We also define: $\omega \wedge \eta$ as $(\omega \wedge \eta)(p) := \omega(p) \wedge \eta(p)$

$f^*\omega$ as $(f^*\omega)(p) := (df_p)^*\omega(p)$

$f : N \longrightarrow M$ smooth

Basis elements:



basis of T_pM : $(\partial_1, \partial_2, \dots, \partial_n)$ with $\partial_j := \varphi_*(e_j) = d\varphi_{h(p)}(e_j)$

basis of $(T_pM)^* = \mathrm{Alt}^1(T_pM)$: $(dx_p^1, dx_p^2, \dots, dx_p^n)$

defined by: $dx_p^j(\partial_k) = \delta^j_k = \begin{cases} 1 , & j = k \\ 0 , & j \neq k \end{cases}$

Proposition: A basis of $\mathrm{Alt}^k(T_pM)$ is given by:

$$\left(dx_p^{\mu_1} \wedge dx_p^{\mu_2} \wedge \dots \wedge dx_p^{\mu_k}\right)_{\mu_1 < \mu_2 < \dots < \mu_k}$$

Example: $\dim(M) = 3$, $\mathrm{Alt}^2(T_pM)$:

$$\left(dx_p^1 \wedge dx_p^2 ,\ dx_p^1 \wedge dx_p^3 ,\ dx_p^2 \wedge dx_p^3 \right)$$

Conclusion: Each k-form on M can locally be written as:

$$\omega(p) = \sum_{\mu_1 < \cdots < \mu_k} \omega_{\mu_1, \mu_2, \ldots, \mu_k}(p) \cdot dx_p^{\mu_1} \wedge dx_p^{\mu_2} \wedge \cdots \wedge dx_p^{\mu_k}$$

$$\omega_{\mu_1, \mu_2, \ldots, \mu_k} : U \longrightarrow \mathbb{R} \qquad \text{component functions}$$

Definition:
- If all component functions are differentiable at p, then ω is differentiable at p.
- If ω is differentiable at all $p \in M$, then ω is called a differential form on M.

$$\omega \in \Omega^k(M)$$

$$\Omega^0(M) := C^\infty(M)$$