The Bright Side of Mathematics

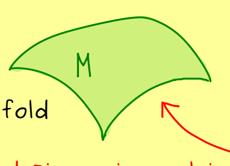
Manifolds – Part 34

Riemannian metric: $g: p \mapsto g_p$ ← inner product on T_pM

(↑ smooth)

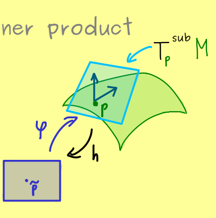
Submanifolds in $\mathbb{R}^N$:

n-dimensional submanifold



$\subseteq \mathbb{R}^N$ ← standard inner product

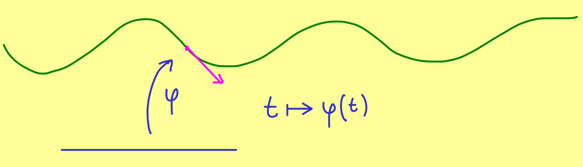
standard Riemannian metric



Note: $T_pM \cong T_p^{sub}M = \text{Span}\left(\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_n}\right)$

$$g^{(h)}_{ij}(p) = \left\langle \frac{\partial \varphi}{\partial x_i}(\tilde{p}), \frac{\partial \varphi}{\partial x_j}(\tilde{p}) \right\rangle_{\text{standard}}$$

Examples: (a) 1-dimensional submanifold in $\mathbb{R}^N$



$t \mapsto \varphi(t)$

$$g^{(h)}_{11}(p) = \langle \varphi'(t), \varphi'(t) \rangle_{\text{standard}} = \|\varphi'(t)\|^2_{\text{standard}}$$

length: $\int_a^b \|\varphi'(t)\|_{\text{standard}}\, dt = \int_a^b \sqrt{\det(G)}\, dt$

(b) $S^2 \subseteq \mathbb{R}^3$ has parameterization given by spherical coordinates:

$$\Phi(\theta, \varphi) = \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix}$$

$\Longrightarrow$ two tangent vectors: $\frac{\partial \Phi}{\partial \theta} = \begin{pmatrix} \cos(\theta)\cos(\varphi) \\ \cos(\theta)\sin(\varphi) \\ -\sin(\theta) \end{pmatrix}$

$\frac{\partial \Phi}{\partial \varphi} = \begin{pmatrix} -\sin(\theta)\sin(\varphi) \\ \sin(\theta)\cos(\varphi) \\ 0 \end{pmatrix}$

$$\Longrightarrow \quad G = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix} \rightsquigarrow \sqrt{\det(G)} = |\sin(\theta)|$$

volume form: $\sqrt{\det(G)}\, d\theta \wedge d\varphi$