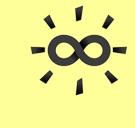
ON STEADY

The Bright Side of Mathematics



Manifolds - Part 28

Wedge product:
$$\Lambda$$
 multiplication defined for $\alpha \in Alt^{k}(V)$, $\beta \in Alt^{s}(V)$

$$\wedge : \operatorname{Alt}^{k}(V) \times \operatorname{Alt}^{s}(V) \longrightarrow \operatorname{Alt}^{k+s}(V) \\ (\alpha, \beta) \longmapsto \alpha \wedge \beta \\ \underbrace{(k+s)-\text{linear}}_{(\alpha \wedge \beta)(V_{1}, \dots, V_{k+s})} :\neq \alpha(V_{1}, \dots, V_{k}) \cdot \beta(V_{k+1}, \dots, V_{k+s}) \\ \text{not a possible definition:} \\ (\text{not alternating})$$

<u>Definition</u>: For $\propto \in Alt^{k}(V)$, $\beta \in Alt^{s}(V)$, we define $\propto \land \beta \in Alt^{k+s}(V)$ by:

$$(\propto \land \beta)(v_{1}, \dots, v_{k+s}) := \frac{1}{k! \cdot s!} \sum_{\sigma \in S_{k+s}} \operatorname{sgn}(\sigma) \propto (v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+s)})$$

Examples: (a)
$$\propto, \beta \in Alt^{4}(V) = V^{*}$$
:

$$(\propto \land \beta)(u, V) = \alpha(u)\beta(V) - \alpha(V)\beta(u)$$
(b) $\propto, \beta \in Alt^{4}(\mathbb{R}^{3}), \alpha(\begin{pmatrix} x_{1} \\ x_{2} \end{pmatrix}) = x_{1}, \beta(\begin{pmatrix} x_{1} \\ x_{2} \end{pmatrix}) = x_{2} = (0, 1, 0)\begin{pmatrix} x_{1} \\ x_{2} \end{pmatrix}$

$$(\propto \land \beta)\begin{pmatrix} \begin{pmatrix} x_{1} \\ x_{2} \end{pmatrix}, \begin{pmatrix} y_{1} \\ y_{2} \end{pmatrix} = x_{1}Y_{2} - y_{4}x_{2} = \langle \begin{pmatrix} x_{1} \\ x_{2} \end{pmatrix}, \begin{pmatrix} 0 \ 1 \ 0 \ 0 \end{pmatrix} \begin{pmatrix} y_{1} \\ y_{2} \end{pmatrix} \rangle$$

$$(dentified with \ \alpha \land \beta$$

Properties:

- (a) $\alpha \wedge \beta = (-1)^{k \cdot s} \beta \wedge \alpha$ (anticommutative) (b) $(\alpha + \alpha') \wedge \beta = \alpha \wedge \beta + \alpha' \wedge \beta$ $(\lambda \alpha) \wedge \beta = \lambda (\alpha \wedge \beta)$ (bilinear)
 - (c) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$ (associative)
 - (d) For a linear map $f: W \to V$ and $\propto \in Alt^{k}(V)$ define: pullback $(f^{*}_{\alpha})(W_{1},...,W_{k}) := \alpha(f(W_{1}),...,f(W_{k}))$ ("natural") $f^{*}(\alpha \wedge \beta) = f^{*}_{\alpha} \wedge f^{*}_{\beta}$