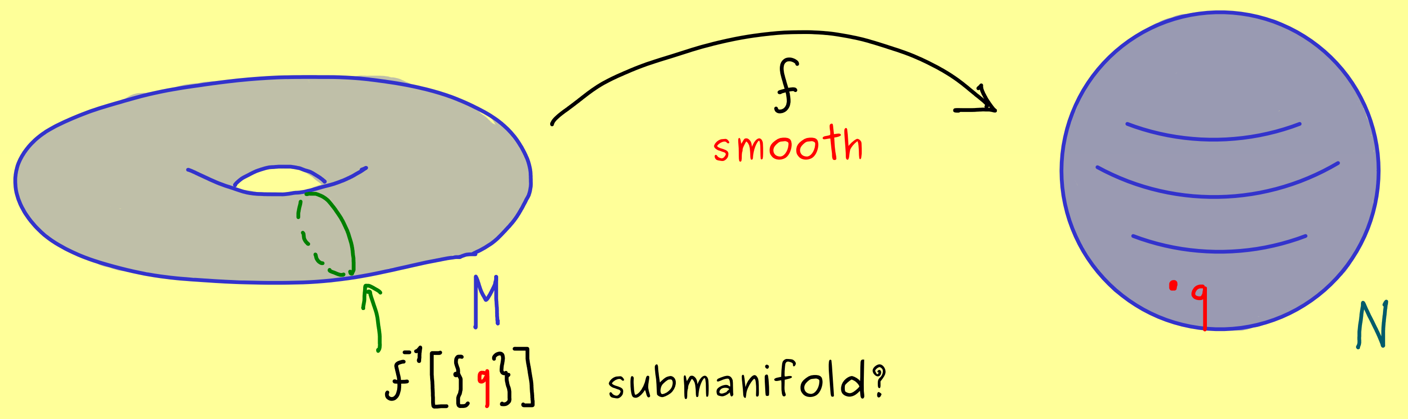
The Bright Side of Mathematics

Manifolds – Part 18

Regular Value Theorem:



Let M, N be smooth manifolds of dimension m and n $(m \geq n)$,
$f: M \longrightarrow N$ be a smooth map, and $q \in N$ be a regular value of f.

↪ $f^{-1}[\{q\}]$ does not contain critical points

↪ $p \in M$ is called a critical point of f if

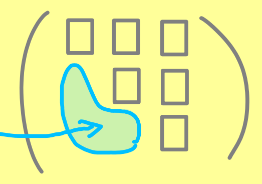
$\operatorname{rank} f_p := \operatorname{rank}\left(J_{k \circ f \circ h^{-1}}(h(p))\right)$

is less than n (not maximal!).

Then: $f^{-1}[\{q\}]$ is a $(m-n)$-dim submanifold of M.

Example: (a) $GL(d, \mathbb{R}) := \{ A \in \mathbb{R}^{d \times d} \mid \det(A) \neq 0 \}$ is manifold of dimension d^2.

(b) $Sym(d \times d, \mathbb{R}) := \{ B \in \mathbb{R}^{d \times d} \mid B^T = B \}$ is manifold of dimension $\frac{d(d+1)}{2}$



$\frac{d^2-d}{2}$ $\qquad$ $d^2 - \frac{d^2-d}{2}$ $\;\Vert$

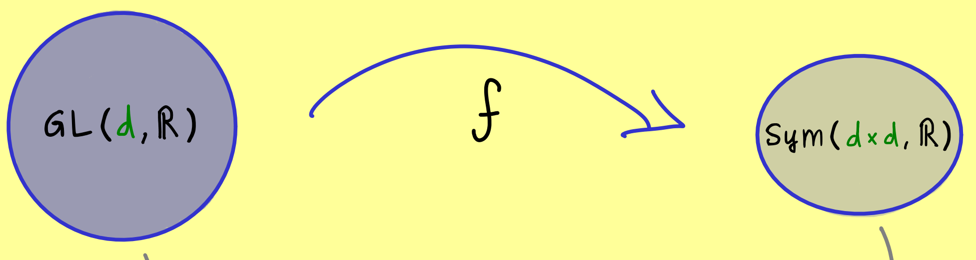
(c) $O(d, \mathbb{R}) := \{ A \in GL(d, \mathbb{R}) \mid A^T A = \mathbb{1} \}$ is a submanifold of $GL(d, \mathbb{R})$

Proof: $f: GL(d, \mathbb{R}) \longrightarrow Sym(d \times d, \mathbb{R})$, $f(A) = A^T A$

Two things to show: (1) $f^{-1}[\{\mathbb{1}\}] = O(d, \mathbb{R})$

(2) $\mathbb{1}$ is a regular value of f

Case $d = 2$:



$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \begin{tikzcd} GL(d,\mathbb{R}) \arrow[r, "f"] \arrow[d, "h"] & Sym(d \times d, \mathbb{R}) \arrow[d, "k"] \\ \mathbb{R}^4 \arrow[r, "k \circ f \circ h^{-1}"] & \mathbb{R}^3 \end{tikzcd} \qquad \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$(k \circ f \circ h^{-1})\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = (k \circ f)\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = k\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}^T \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right)$$

$$= k\left(\begin{pmatrix} x_1^2 + x_3^2 & x_1 x_2 + x_3 x_4 \\ x_1 x_2 + x_3 x_4 & x_2^2 + x_4^2 \end{pmatrix} \right) = \begin{pmatrix} x_1^2 + x_3^2 \\ x_1 x_2 + x_3 x_4 \\ x_2^2 + x_4^2 \end{pmatrix}$$

Jacobian matrix: $$J_{k \circ f \circ h^{-1}}(x) = \begin{pmatrix} 2x_1 & 0 & 2x_3 & 0 \\ x_2 & x_1 & x_4 & x_3 \\ 0 & 2x_2 & 0 & 2x_4 \end{pmatrix}$$

rank = 3? Not for: $x_1 = x_2 = 0$
$x_3 = x_4 = 0$
$x_1 = x_3 = 0$
$x_2 = x_4 = 0$

If $f(A) = \mathbb{1} \Rightarrow J_{k \circ f \circ h^{-1}}(h(A))$ has rank 3 $\Rightarrow \mathbb{1}$ regular value

$\Rightarrow O(d, \mathbb{R})$ is a submanifold of dimension $d^2 - \frac{d(d+1)}{2} = \frac{d(d-1)}{2}$