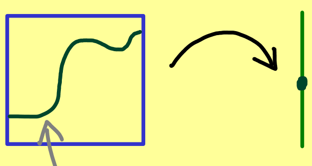
Manifolds – Part 15

Regular value theorem in $\mathbb{R}^n$ = preimage theorem = submersion theorem

$f: \mathbb{R}^n \to \mathbb{R}^m$ smooth



preimage = smooth submanifold?

Definition: $f: U \to \mathbb{R}^m$, $U \subseteq \mathbb{R}^n$ open, C^1-function.

(1) $x \in U$ is called a critical point of f if

df_x is not surjective (or $J_f(x)$ has rank less than m)

(2) $c \in f[U]$ is called a regular value of f if

$f^{-1}[\{c\}]$ does not contain any critical points.

Theorem: $f: U \to \mathbb{R}^m$, $U \subseteq \mathbb{R}^n$ open, C^∞- function. $(n \geq m)$

If c is a regular value of f, then

$f^{-1}[\{c\}]$ is an $(n-m)$-dimensional submanifold of $\mathbb{R}^n$.

Proof: Use implicite function theorem.

Example: $f: \mathbb{R}^n \to \mathbb{R}$, $f(x_1, \dots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2$

$$J_f(x_1, \dots, x_n) = (2x_1 \;\; 2x_2 \;\; \cdots \;\; 2x_n)$$

$\Rightarrow$ $x = 0$ is the only critical point.

Hence: 1 is a regular value.

$\Rightarrow$ $f^{-1}[\{1\}] = S^{n-1}$ submanifold of $\mathbb{R}^n$.