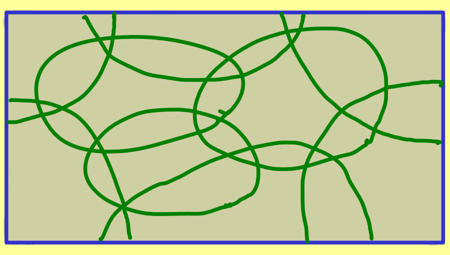
Manifolds – Part 8

$[a,b] \subseteq \mathbb{R}$ compact (Bolzano–Weierstrass and Heine–Borel)

$(X, \mathcal{T})$



cover with open sets

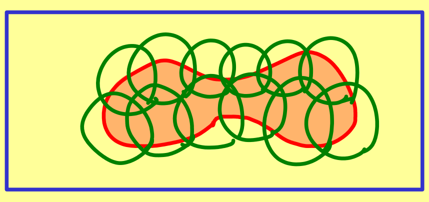
$\Downarrow$

do finitely many suffice?

Definition: Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$.

A is called compact if

$$\bigcup_{i \in I} U_i \supseteq A \text{ with } U_i \in \mathcal{T} \implies \text{there is a finite } I_0 \subseteq I \text{ with: } \bigcup_{i \in I_0} U_i \supseteq A$$



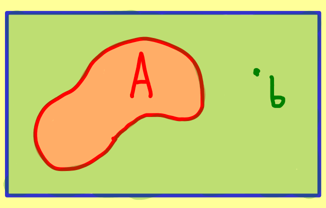
We know: $A \subseteq \mathbb{R}^n$ compact $\iff$ A closed and bounded (Heine–Borel theorem)

with standard topology

Proposition: Let $(X, \mathcal{T})$ be a Hausdorff space. Then:

$A \subseteq X$ compact $\implies$ A closed $\left(\begin{array}{l} X \setminus A \text{ open} \\ X \setminus A \in \mathcal{T} \end{array} \right)$

Proof:

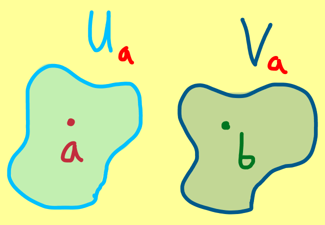


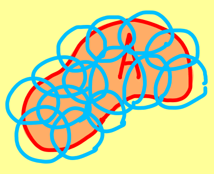
X

Assume A is compact.

Fix $b \in X \setminus A$.

For any $a \in A$, there are $U_a, V_a \in \mathcal{T}$ with $a \in U_a$, $b \in V_a$ and $U_a \cap V_a = \emptyset$





$A \subseteq \bigcup_{a \in A} U_a$ (open cover)

A compact

$\implies A \subseteq \bigcup_{j=1}^{n} U_{a_j}$ (finite subcover)

$V_{a_1} \cdots V_{a_n}$

$\implies V := \bigcap_{j=1}^{n} V_{a_j}$ open neighbourhood of b

with $A \cap V \subseteq \bigcup_{j=1}^{n} U_{a_j} \cap \bigcap_{j=1}^{n} V_{a_j} = \emptyset$

$\implies b$ is an interior point of $X \setminus A \implies A$ closed