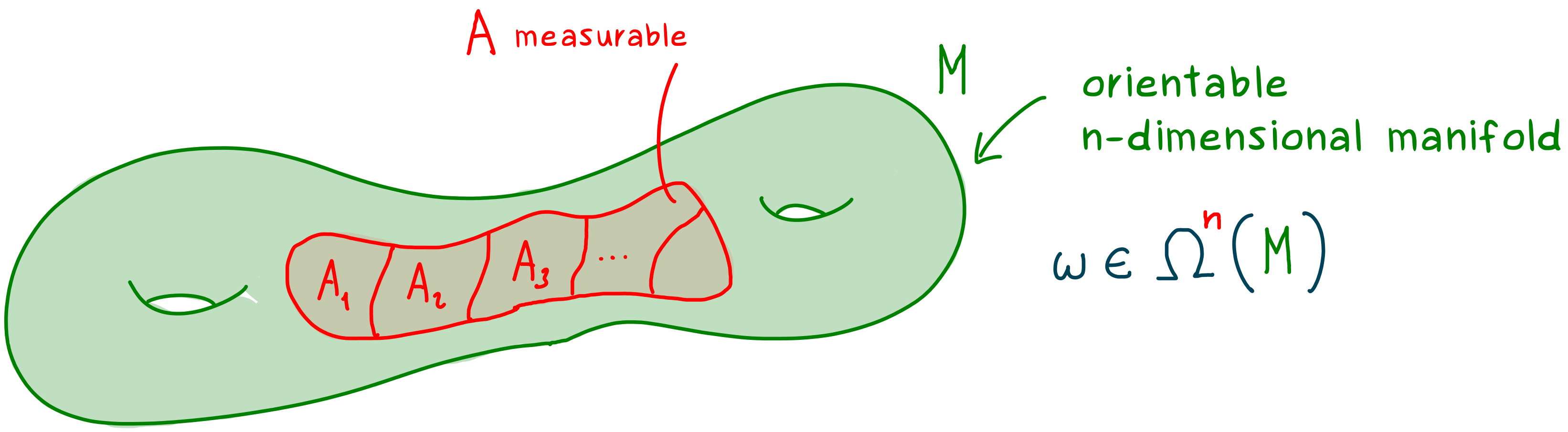
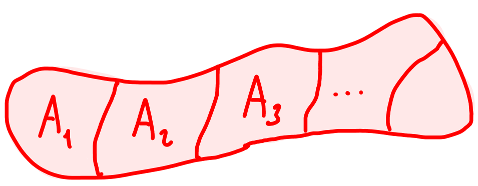
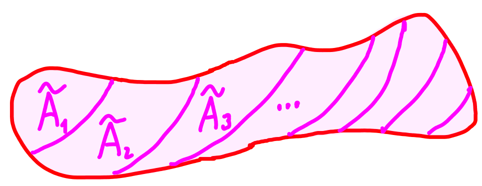
# Manifolds - Part 43




$\omega \in \Omega^n(M)$

$$\int_A \omega := \sum_{k=1}^{\infty} \int_{A_k} \omega \overset{\text{proof?}}{=} \sum_{m=1}^{\infty} \int_{\tilde{A}_m} \omega$$

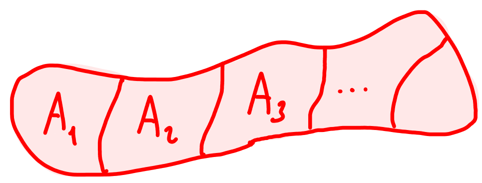



**Proposition:** (well-definedness of $\int_A \omega$)

$(U_k, h_k)_{k\in\mathbb{N}}$ atlas, $A = \bigcup_{k\in\mathbb{N}} A_k$ disjoint $A_k \subseteq U_k$ with:

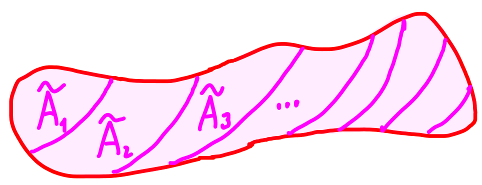
(1) $\int_{A_k} \omega$ exists for all $k \in \mathbb{N}$




(2) $$\sum_{k=1}^{\infty} \int_{h_k[A_k]} \left| \omega_{1,2,\dots,n}\left(h_k^{-1}(x)\right) \right| \, d^n x < \infty$$

$(\tilde{U}_m, \tilde{h}_m)_{m\in\mathbb{N}}$ atlas, $A = \bigcup_{m\in\mathbb{N}} \tilde{A}_m$ disjoint $\tilde{A}_m \subseteq \tilde{U}_m$ .

(measurable)




**Then:**

($\tilde{1}$) $\int_{\tilde{A}_m} \omega$ exists for all $m \in \mathbb{N}$

($\tilde{2}$) $$\sum_{m=1}^{\infty} \int_{\tilde{h}_m[\tilde{A}_m]} \left| \omega_{1,2,\dots,n}\left(\tilde{h}_m^{-1}(x)\right) \right| \, d^n x < \infty$$

**and:**
$$\sum_{m=1}^{\infty} \int_{\tilde{h}_m[\tilde{A}_m]} \omega_{1,2,\dots,n}\big(\tilde{h}_m^{-1}(x)\big)\; d^n x = \sum_{k=1}^{\infty} \int_{h_k[A_k]} \omega_{1,2,\dots,n}\big(h_k^{-1}(x)\big)\; d^n x = \int_A \omega$$

**Proof:**


new decomposition: $\quad A = \bigcup_{k,m} (A_k \cap \hat{A}_m)$

$$\int_{h_k[A_k \cap \hat{A}_m]} \big|\omega_{1,2,\dots,n}\big(h_k^{-1}(x)\big)\big|\, d^n x \overset{\text{part } 40}{=} \int_{\tilde{h}_m[A_k \cap \hat{A}_m]} \big|\omega_{1,2,\dots,n}\big(\tilde{h}_m^{-1}(x)\big)\big|\, d^n x$$

$$\Rightarrow \sum_{m=1}^{\infty} \int_{h_k[A_k \cap \hat{A}_m]} \big|\omega_{1,2,\dots,n}\big(h_k^{-1}(x)\big)\big|\, d^n x = \sum_{m=1}^{\infty} \int_{\tilde{h}_m[A_k \cap \hat{A}_m]} \big|\omega_{1,2,\dots,n}\big(\tilde{h}_m^{-1}(x)\big)\big|\, d^n x$$

$$\| $$

$$\int_{\bigcup_{m\in\mathbb{N}} h_k[A_k \cap \hat{A}_m]} \big|\omega_{1,2,\dots,n}\big(h_k^{-1}(x)\big)\big|\, d^n x$$

$$= \int_{h_k[A_k]} \big|\omega_{1,2,\dots,n}\big(h_k^{-1}(x)\big)\big|\, d^n x$$

finite!

$$\Rightarrow \sum_{k=1}^{\infty} \int_{h_k[A_k]} \big|\omega_{1,2,\dots,n}\big(h_k^{-1}(x)\big)\big|\, d^n x = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \int_{\tilde{h}_m[A_k \cap \hat{A}_m]} \big|\omega_{1,2,\dots,n}\big(\tilde{h}_m^{-1}(x)\big)\big|\, d^n x$$

$$= \sum_{m=1}^{\infty} \int_{\tilde{h}_m[\hat{A}_m]} \big|\omega_{1,2,\dots,n}\big(\tilde{h}_m^{-1}(x)\big)\big|\, d^n x$$

same calculation without absolute value

$$\Rightarrow \sum_{k=1}^{\infty} \int_{h_k[A_k]} \omega_{1,2,\dots,n}\big(h_k^{-1}(x)\big)\; d^n x = \sum_{m=1}^{\infty} \int_{\tilde{h}_m[\tilde{A}_m]} \omega_{1,2,\dots,n}\big(\tilde{h}_m^{-1}(x)\big)\; d^n x$$

$\square$