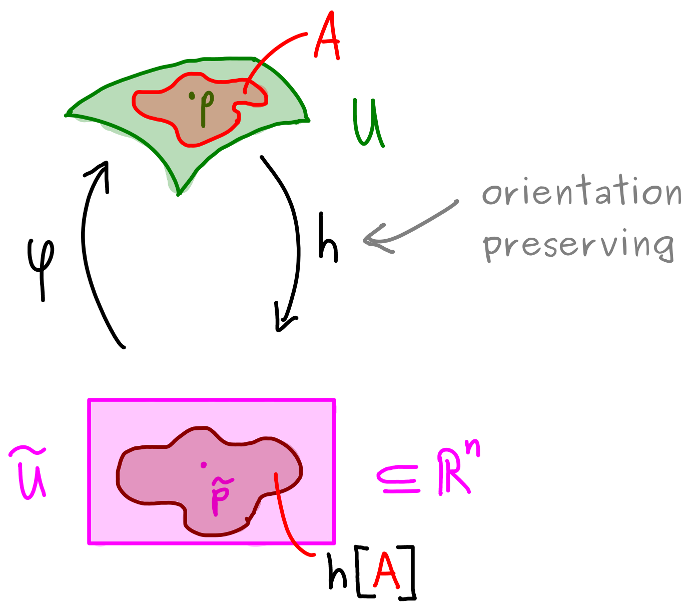
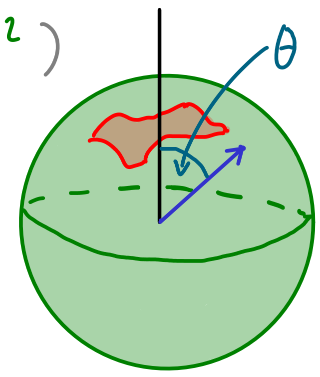
Manifolds - Part 41

We already know:

$$\int_A \omega := \int_{h[A]} \varphi^* \omega$$



Example: ω canonical volume form on S^2 (measures areas on S^2)



$$\Phi : \underbrace{(0,\pi) \times (0, 2\pi)}_{= \tilde{U}} \longrightarrow \mathbb{R}^3$$

$$(\theta, \varphi) \longmapsto \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix}$$

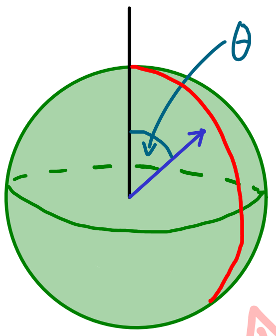
$$\int_{\Phi[\tilde{U}]} \omega = \int_{\tilde{U}} \Phi^* \omega$$

canonical volume form: $\omega(p) = \underbrace{\sqrt{\det(G(p))}}_{\sin(\theta)} \; \underbrace{dx^1_p}_{d\theta} \wedge \underbrace{dx^2_p}_{d\varphi}$

for $p = \Phi(\theta, \varphi)$ 1-forms on S^2

$$(\Phi^* \omega)(\tilde{p}) = \sin(\theta) \cdot \underbrace{\det(\cdot, \cdot)}_{d\theta \wedge d\varphi}, \quad \tilde{p} = \begin{pmatrix} \theta \\ \varphi \end{pmatrix}$$

1-forms on $\tilde{U} \subseteq \mathbb{R}^2$



in short: $\omega = \sin(\theta)\; d\theta \wedge d\varphi$

$$\Phi^* \omega = \sin(\theta)\; d\theta \wedge d\varphi$$

$$\int_{S^2 \setminus \{\dots\}} \omega = \int_{\Phi[\tilde{U}]} \omega = \int_{(0,\pi)\times(0,2\pi)} \Phi^* \omega = \int_{(0,\pi)\times(0,2\pi)} \sin(\theta)\; d\theta \wedge d\varphi$$

null set

$$= \int_0^{\pi} \left(\int_0^{2\pi} \sin(\theta)\; d\varphi \right) d\theta = 4\pi$$

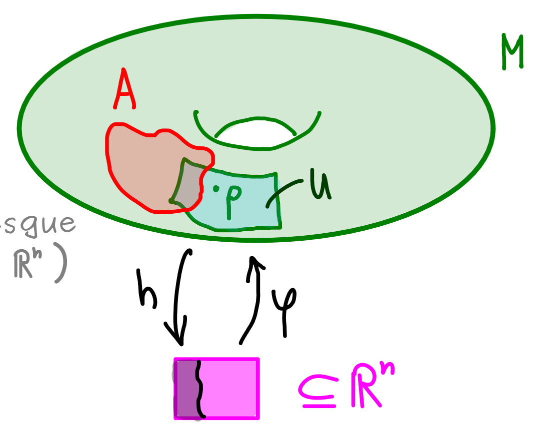
Definition: Let M be an orientable n-dimensional manifold and $\omega \in \Omega^n(M)$.

A set $A \subseteq M$ is called

- measurable if $h[A \cap U]$ is measurable for every chart (U, h).

  (w.r.t. Lebesgue measure in $\mathbb{R}^n$)

- null set (set with measure zero) if $h[A \cap U]$ has Lebesgue measure 0 for every chart (U, h).



We get: $\int_A \omega$ is defined for every measurable set $A \subseteq U$ (where (U,h) is a chart)

(assuming $\int_{h[A]} \varphi^* \omega$ exists in $\mathbb{R}^n$)

and $\int_B \omega := \int_{B \setminus N} \omega$ if $B \setminus N \subseteq U$ (where (U,h) is a chart) and N is a null set.

Hence: $\int_{S^2} \omega = 4\pi$