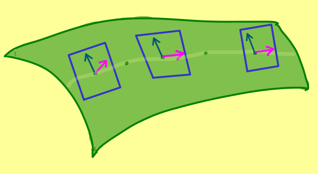
Manifolds – Part 32



orientable manifold M

Fact: Let M be an n-dim smooth manifold. Then the following claims are equivalent:

(a) M is orientable : We have $\{ (T_p M, or_p) \}$ such that

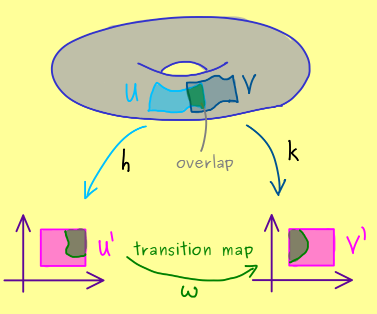
$$\forall p \in M \;\; \exists (U, h) \;\; \forall x \in U : \quad \left(\partial_1^{(h)}(x), \partial_2^{(h)}(x), \dots, \partial_n^{(h)}(x) \right) \in or_x$$

(b) There is an atlas for M collection of charts that cover the manifold

such that all transition maps

$\omega :$ → satisfy:

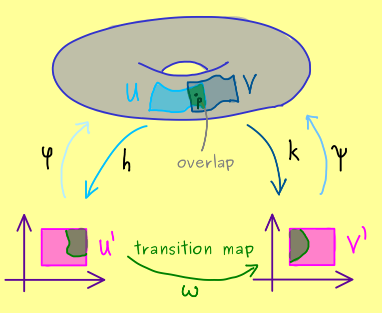
$$\det(J_\omega(x)) > 0$$

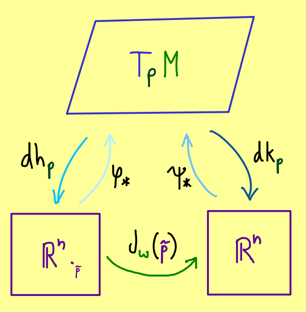


(c) There is a differential form (volume form)

$$\omega \in \Omega^n(M) \quad \text{with} \quad \omega(p) \neq 0 \quad \text{for all } p \in M.$$

Proof: (a) $\Longleftrightarrow$ (b)





We have: $\psi_*(\underbrace{J_\omega(\tilde{p})\, e_1}_{\text{first column of Jacobian}}) = \varphi_*(e_1)$

first column of Jacobian $= \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \sum_j \lambda_j e_j$

$$\Longrightarrow \sum_{j=1}^{n} \lambda_j \underbrace{\psi_*(e_j)}_{\partial_j^{(k)}(p)} = \underbrace{\varphi_*(e_1)}_{\partial_1^{(h)}(p)} \quad (*)$$

Change-of-basis matrix: $\mathcal{B} = \left(\partial_1^{(h)}(p), \dots, \partial_n^{(h)}(p) \right) \xrightarrow[T_{\mathcal{C} \leftarrow \mathcal{B}}]{} \mathcal{C} = \left(\partial_1^{(k)}(p), \dots, \partial_n^{(k)}(p) \right)$

$$\overset{(*)}{\Longrightarrow} \quad T_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} \lambda_1 & \cdots \\ \lambda_2 & \\ \vdots & \\ \lambda_n & \end{pmatrix} = J_\omega(\tilde{p})$$

Hence:

$$\det(T_{\mathcal{C} \leftarrow \mathcal{B}}) > 0 \quad \Longleftrightarrow \quad \det(J_\omega(x)) > 0$$

$$(a) \quad \Longleftrightarrow \quad (b)$$