Manifolds – Part 29

M smooth manifold of dimension n $\Longrightarrow$ T_pM n-dimensional vector space

Definition:

$$\omega : M \longrightarrow \bigcup_{p \in M} \mathrm{Alt}^k(T_pM)$$
$$p \longmapsto \omega_p = \omega(p) \in \mathrm{Alt}^k(T_pM)$$

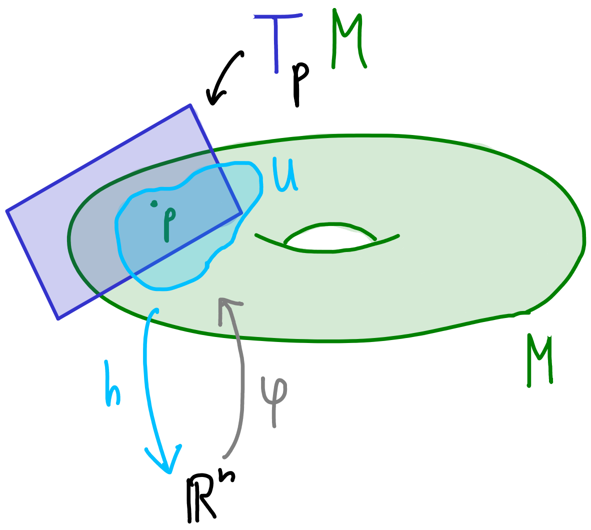
is called a k-form on M.

We also define: $\omega \wedge \eta$ as $(\omega \wedge \eta)(p) := \omega(p) \wedge \eta(p)$

$f^* \omega$ as $(f^*\omega)(p) := (df_p)^* \omega(f(p))$

$f : N \longrightarrow M$ smooth

Basis elements:



basis of T_pM : $(\partial_1, \partial_2, \dots, \partial_n)$ with $\partial_j := \varphi_*(e_j) = d\varphi_{h(p)}(e_j)$

basis of $(T_pM)^* = \mathrm{Alt}^1(T_pM)$: $(dx_p^1, dx_p^2, \dots, dx_p^n)$

defined by: $dx_p^j(\partial_k) = \delta^j_k = \begin{cases} 1 , & j = k \\ 0 , & j \neq k \end{cases}$

Proposition: A basis of $\mathrm{Alt}^k(T_pM)$ is given by:

$$\left(dx_p^{\mu_1} \wedge dx_p^{\mu_2} \wedge \dots \wedge dx_p^{\mu_k} \right)_{\mu_1 < \mu_2 < \dots < \mu_k}$$

Example: $\dim(M) = 3$, $\mathrm{Alt}^2(T_pM)$:

$$\left(dx_p^1 \wedge dx_p^2 ,\ dx_p^1 \wedge dx_p^3 ,\ dx_p^2 \wedge dx_p^3 \right)$$

Conclusion: Each k-form on M can locally be written as:

$$\omega(p) = \sum_{\mu_1 < \dots < \mu_k} \omega_{\mu_1, \mu_2, \dots, \mu_k}(p) \cdot dx_p^{\mu_1} \wedge dx_p^{\mu_2} \wedge \dots \wedge dx_p^{\mu_k}$$

$\omega_{\mu_1, \mu_2, \dots, \mu_k} : U \longrightarrow \mathbb{R}$ component functions

Definition:

- If all component functions are differentiable at p, then ω is differentiable at p.
- If ω is differentiable at all $p \in M$, then ω is called a differential form on M.

$\omega \in \Omega^k(M)$

$\Omega^0(M) := C^\infty(M)$