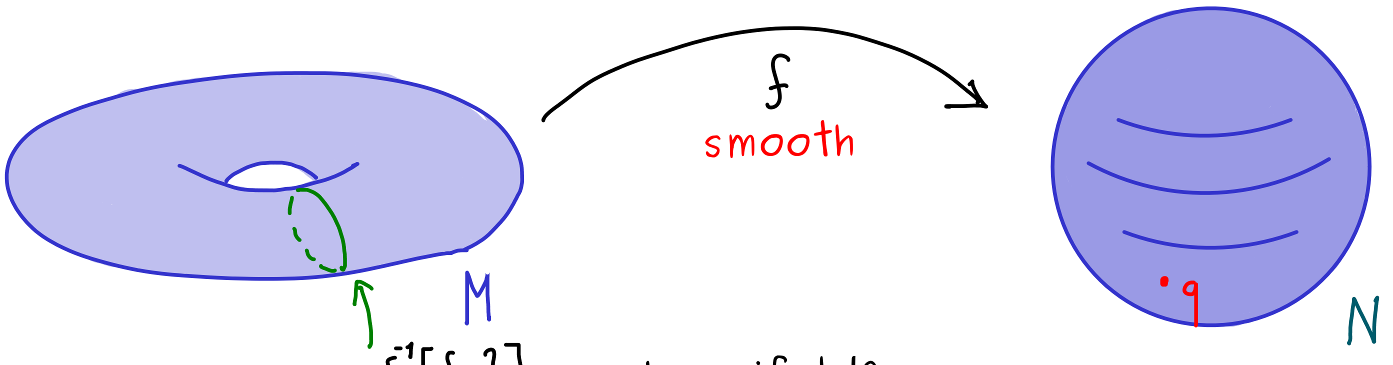
# Manifolds – Part 18

## Regular Value Theorem:




$f^{-1}[\{q\}]$ submanifold?

Let $M, N$ be smooth manifolds of dimension $m$ and $n$ $(m \geq n)$,
$f: M \longrightarrow N$ be a smooth map, and $q \in N$ be a regular value of $f$.

↪ $f^{-1}[\{q\}]$ does not contain critical points

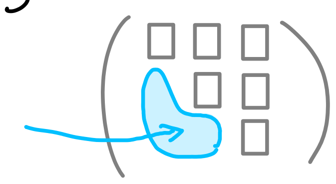
↪ $p \in M$ is called a critical point of $f$ if
$$\operatorname{rank} f_p := \operatorname{rank}\left(J_{k \circ f \circ h^{-1}}(h(p))\right)$$
is less than $n$ (not maximal!).

Then: $f^{-1}[\{q\}]$ is a $(m-n)$-dim submanifold of $M$.

**Example:**

(a) $GL(d, \mathbb{R}) := \left\{ A \in \mathbb{R}^{d \times d} \mid \det(A) \neq 0 \right\}$ is manifold of dimension $d^2$.

(b) $\operatorname{Sym}(d \times d, \mathbb{R}) := \left\{ B \in \mathbb{R}^{d \times d} \mid B^T = B \right\}$ is manifold of dimension $\dfrac{d(d+1)}{2}$

$\dfrac{d^2 - d}{2}$  $d^2 - \dfrac{d^2 - d}{2}$ $=$

(c) $O(d, \mathbb{R}) := \left\{ A \in GL(d, \mathbb{R}) \mid A^T A = \mathbb{1} \right\}$ is a submanifold of $GL(d, \mathbb{R})$

Proof: $f : GL(d,\mathbb{R}) \longrightarrow Sym(d\times d,\mathbb{R})$ , $f(A) = A^T A$

Two things to show: (1) $f^{-1}[\{\mathbb{1}\}] = O(d,\mathbb{R})$

(2) $\mathbb{1}$ is a regular value of $f$

Case $d=2$ :

$$\begin{tikzcd} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \arrow[d, mapsto] & GL(d,\mathbb{R}) \arrow[rr, bend left, "f"] \arrow[d, "h"] & & Sym(d\times d,\mathbb{R}) \arrow[d, "k"] & \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} \arrow[d, mapsto] \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} & \mathbb{R}^4 \arrow[rr, "k\circ f\circ h^{-1}"] & & \mathbb{R}^3 & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{tikzcd}$$

$$(k\circ f\circ h^{-1})\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = (k\circ f)\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = k\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}^T \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right)$$

$$= k\left(\begin{pmatrix} x_1^2 + x_3^2 & x_1 x_2 + x_3 x_4 \\ x_1 x_2 + x_3 x_4 & x_2^2 + x_4^2 \end{pmatrix}\right) = \begin{pmatrix} x_1^2 + x_3^2 \\ x_1 x_2 + x_3 x_4 \\ x_2^2 + x_4^2 \end{pmatrix}$$

Jacobian matrix: $$J_{k\circ f\circ h^{-1}}(x) = \begin{pmatrix} 2x_1 & 0 & 2x_3 & 0 \\ x_2 & x_1 & x_4 & x_3 \\ 0 & 2x_2 & 0 & 2x_4 \end{pmatrix}$$

rank = 3? Not for: $x_1 = x_2 = 0$

$x_3 = x_4 = 0$

$x_1 = x_3 = 0$

$x_2 = x_4 = 0$

If $f(A) = \mathbb{1} \implies J_{k\circ f\circ h^{-1}}(h(A))$ has rank 3 $\implies \mathbb{1}$ regular value

$\implies O(d,\mathbb{R})$ is a submanifold of dimension $d^2 - \frac{d(d+1)}{2} = \frac{d(d-1)}{2}$