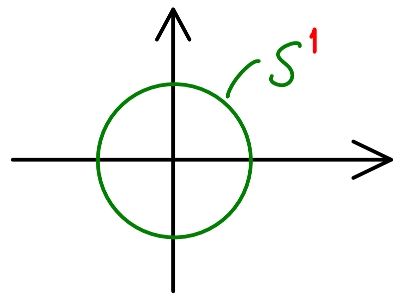
Manifolds – Part 5

$(X, \mathcal{T})$ topological space $\rightsquigarrow$ $(X/\sim, \hat{\mathcal{T}})$ quotient space

Projective space: $\mathbb{P}^n(\mathbb{R}) =$ set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$

$S^n \subseteq \mathbb{R}^{n+1}$

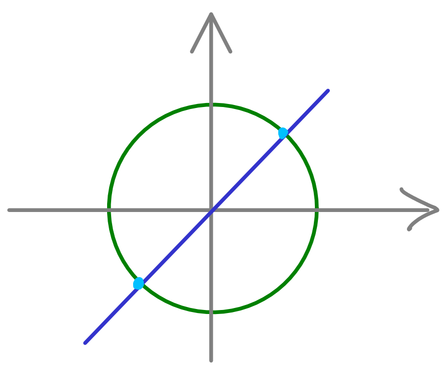


$$S^n := \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$$

Euclidean norm

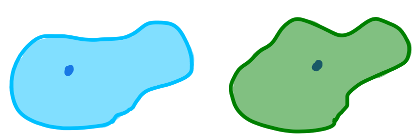
equivalence relation: $x \sim -x$

Let's define: $x \sim y \;:\Longleftrightarrow\; (x = y \text{ or } x = -y)$



$$\mathbb{P}^n(\mathbb{R}) := S^n/\sim \quad \text{with quotient topology}$$

Is $\mathbb{P}^n(\mathbb{R})$ a Hausdorff space?



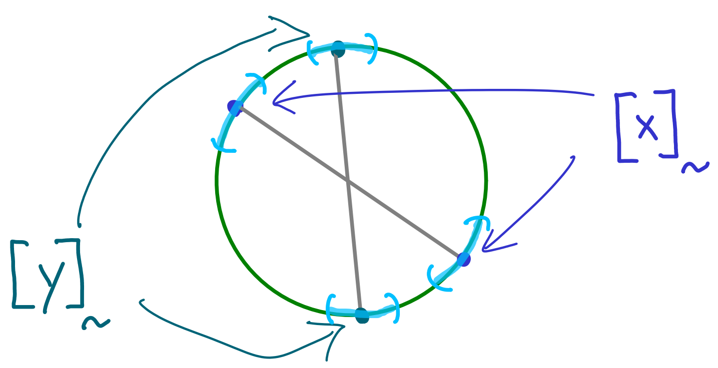
Take $[x]_\sim, [y]_\sim \in \mathbb{P}^n(\mathbb{R})$ with $[x]_\sim \neq [y]_\sim \Longrightarrow x \neq y$ and $x \neq -y$

Take open neighbourhoods

$U, V \subseteq S^n$ of x and y, respectively,

with $U \cap V = \emptyset$, $-U \cap V = \emptyset$

$-U \cap -V = \emptyset$, $U \cap -V = \emptyset$



Look at: $\hat{U} := q[U]$, $\quad q: S^n \to S^n/\sim$ canonical projection

$$q^{-1}[\hat{U}] = \underbrace{U \cup (-U)}_{\text{open}} \in \mathcal{T} \quad \Rightarrow \quad \underbrace{\hat{U}}_{\text{open}} \in \hat{\mathcal{T}}$$

(the same for $\hat{V} := q[V]$)

We find: $q^{-1}[\hat{U} \cap \hat{V}] = q^{-1}[\hat{U}] \cap q^{-1}[\hat{V}] = (U \cup (-U)) \cap (V \cup -V) = \emptyset$

$$\overset{q \text{ surjective}}{\Longrightarrow} \hat{U} \cap \hat{V} = \emptyset$$