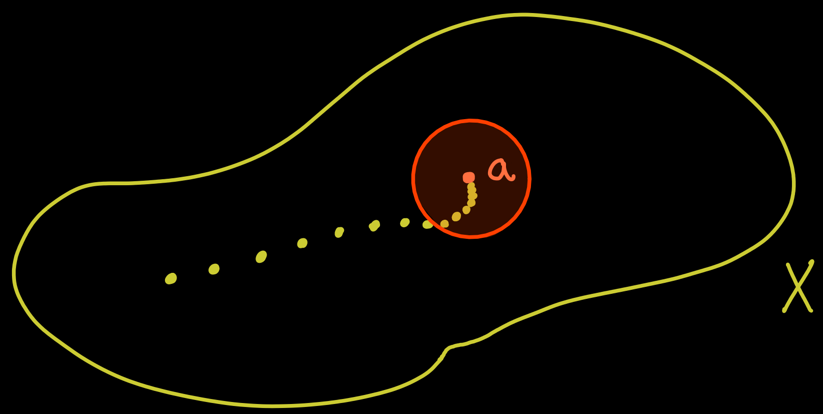
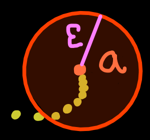
Manifolds – Part 3

$(X, \mathcal{T})$ topological space



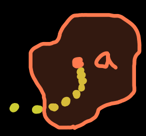
Convergence: $(a_n)_{n\in\mathbb{N}}$, $a_n \in X$

converges to $a \in X$

In a metric space: 

The sequence members lie in each ε-ball around a, eventually.

For each ε-ball $B_\varepsilon(a)$, there is $N \in \mathbb{N}$ such that for all $n \geq N$: $\quad a_n \in B_\varepsilon(a)$

In a topological space: 

$\underbrace{\text{open neighbourhood of } a}_{\text{an open set } U \in \mathcal{T} \text{ with } a \in U}$

Definition: $(X, \mathcal{T})$ topological space, $(a_n)_{n\in\mathbb{N}}$ sequence in X.

$a_n \xrightarrow{n\to\infty} a \quad :\Longleftrightarrow$ For each $U \in \mathcal{T}$ with $a \in U$, there is $N \in \mathbb{N}$ such that for all $n \geq N$: $\quad a_n \in U$

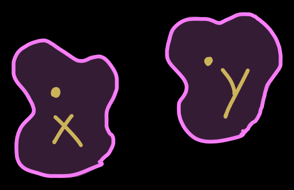
Example: $X = \mathbb{R}$, $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(b, \infty) \mid b \in \mathbb{R}\}$

$$(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$$

- converges to 0: each open neighbourhood of 0 looks like (b, ∞) for $b < 0$, so: $\frac{1}{n} \in (b, \infty)$
- converges to -1: each open neighbourhood of -1 looks like (b, ∞) for $b < -1$, so: $\frac{1}{n} \in (b, \infty)$
- converges to -2

Definition: A topological space $(X, \mathcal{T})$ is called a Hausdorff space if for all $x, y \in X$ with $x \neq y$ there is an open neighbourhood of x: $U_x \in \mathcal{T}$

and there is an open neighbourhood of y: $U_y \in \mathcal{T}$



with: $$U_x \cap U_y = \emptyset$$