The Bright Side of Mathematics

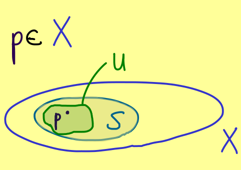
Manifolds – Part 2

$\mathcal{T} \subseteq \mathcal{P}(X)$ topology on X:
(1) $\emptyset, X \in \mathcal{T}$
(2) $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$
(3) $(A_i)_{i \in I}$ with $A_i \in \mathcal{T}$ $\Rightarrow \bigcup_{i \in I} A_i \in \mathcal{T}$

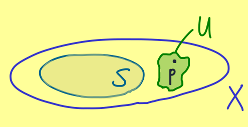
$(X, \mathcal{T})$ is called a topological space.

Important names: $(X, \mathcal{T})$ topological space, $S \subseteq X$, $p \in X$

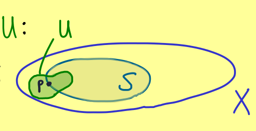
(a) p interior point of S :$\Longleftrightarrow$ There is an open set $U \in \mathcal{T}$: $p \in U$ and $U \subseteq S$



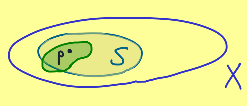
(b) p exterior point of S :$\Longleftrightarrow$ There is an open set $U \in \mathcal{T}$: $p \in U$ and $U \subseteq X \setminus S$



(c) p boundary point of S :$\Longleftrightarrow$ For all open sets $U \in \mathcal{T}$ with $p \in U$: $U \cap S \neq \emptyset$ and $U \cap (X \setminus S) \neq \emptyset$



(d) p accumulation point of S :$\Longleftrightarrow$ For all open sets $U \in \mathcal{T}$ with $p \in U$: $U \setminus \{p\} \cap S \neq \emptyset$



More names:
(a) $S^\circ := \{ p \in X \mid p \text{ interior point of } S \}$ interior of S
(b) $\mathrm{Ext}(S) := \{ p \in X \mid p \text{ exterior point of } S \}$ exterior of S
(c) $\partial S := \{ p \in X \mid p \text{ boundary point of } S \}$ boundary of S
(d) $S' := \{ p \in X \mid p \text{ accumulation point of } S \}$ derived set of S
(e) $\overline{S} := S \cup \partial S$ closure of S

Example: $X = \mathbb{R}$, $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) \mid a \in \mathbb{R}\}$

$S = (0,1)$ ← not an open set!
← no interior points: there is no $\emptyset \neq U \in \mathcal{T}$ with $U \subseteq S$

$\Rightarrow S^\circ = \emptyset$

$X \setminus S = (-\infty, 0] \cup [1, \infty) \Rightarrow \mathrm{Ext}(S) = (1, \infty)$

$\Rightarrow \partial S = (-\infty, 1] \Rightarrow \overline{S} = (-\infty, 1]$