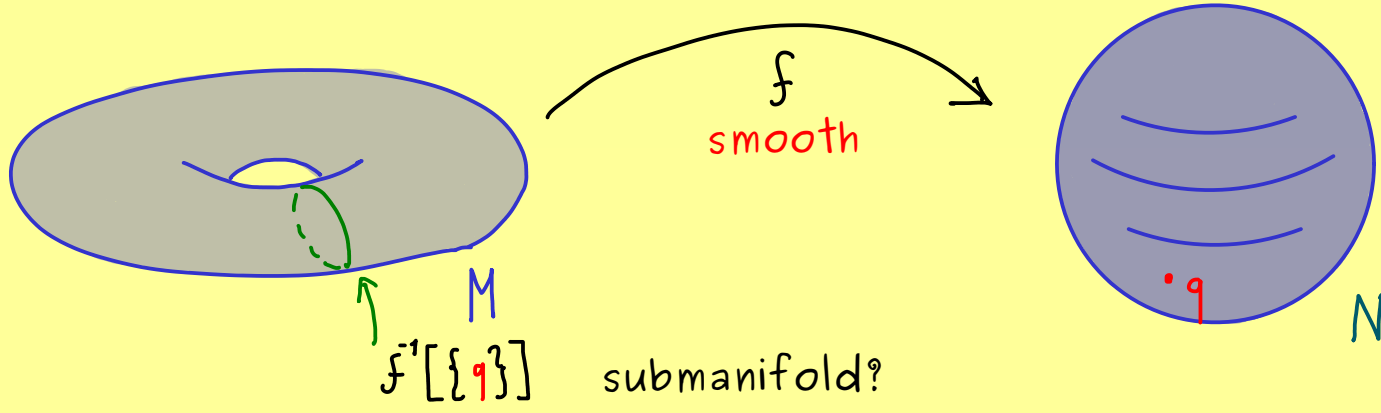




Manifolds - Part 18

Regular Value Theorem:



Let M, N be smooth manifolds of dimension m and n ($m \geq n$),
 $f: M \rightarrow N$ be a smooth map, and $q \in N$ be a regular value of f .

$\hookrightarrow f^{-1}(\{q\})$ does not contain critical points

$\hookrightarrow p \in M$ is called a critical point of f if
 $\text{rank } f_p := \text{rank} \left(J_{k \circ f \circ h^{-1}}(h(p)) \right)$
 is less than n (not maximal!).

Then: $f^{-1}(\{q\})$ is a $(m-n)$ -dim submanifold of M .

Example: (a) $GL(d, \mathbb{R}) := \{A \in \mathbb{R}^{d \times d} \mid \det(A) \neq 0\}$ is manifold of dimension d^2 .

(b) $Sym(d \times d, \mathbb{R}) := \{B \in \mathbb{R}^{d \times d} \mid B^T = B\}$ is manifold of dimension $\frac{d(d+1)}{2}$
 $d, \mathbb{R} \frac{d^2-d}{2} \rightarrow \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix} \quad d^2 - \frac{d^2-d}{2} //$

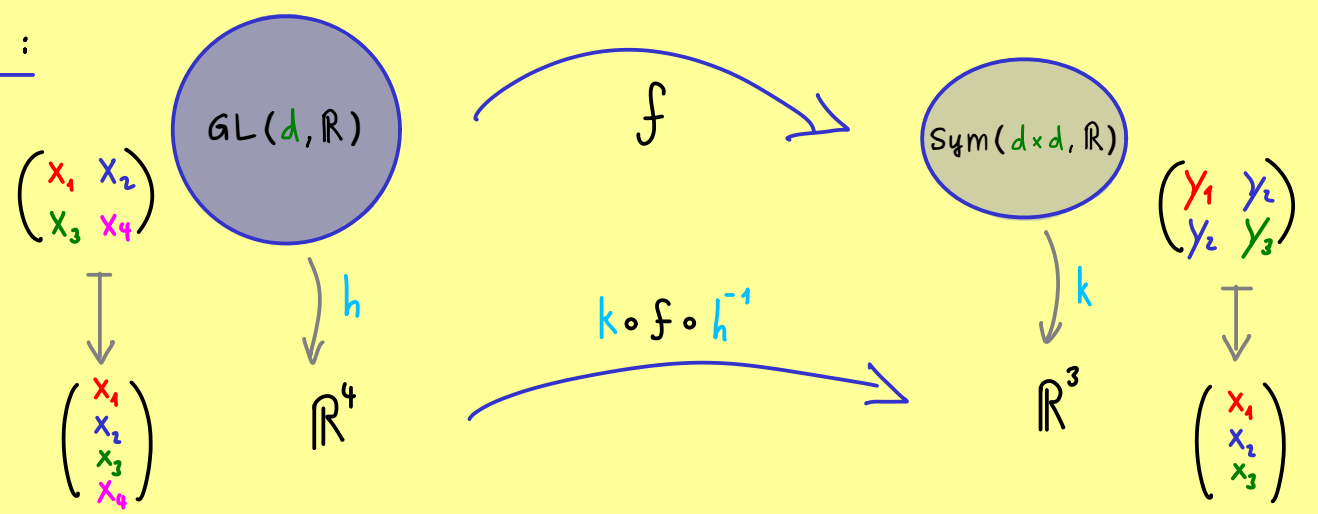
(c) $O(d, \mathbb{R}) := \{A \in GL(d, \mathbb{R}) \mid A^T A = \mathbb{1}\}$ is a submanifold of $GL(d, \mathbb{R})$

Proof: $f: GL(d, \mathbb{R}) \rightarrow Sym(d \times d, \mathbb{R})$, $f(A) = A^T A$

Two things to show: (1) $f^{-1}(\{\mathbb{1}\}) = O(d, \mathbb{R})$

(2) $\mathbb{1}$ is a regular value of f

Case $d=2$:



$$(k \circ f \circ h^{-1}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = (k \circ f) \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = k \left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}^T \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right)$$

$$= k \left(\begin{pmatrix} x_1^2 + x_3^2 & x_1 x_2 + x_3 x_4 \\ x_1 x_2 + x_3 x_4 & x_2^2 + x_4^2 \end{pmatrix} \right) = \begin{pmatrix} x_1^2 + x_3^2 \\ x_1 x_2 + x_3 x_4 \\ x_2^2 + x_4^2 \end{pmatrix}$$

Jacobian matrix: $J_{k \circ f \circ h^{-1}}(x) = \begin{pmatrix} 2x_1 & 0 & 2x_3 & 0 \\ x_2 & x_1 & x_4 & x_3 \\ 0 & 2x_2 & 0 & 2x_4 \end{pmatrix}$

rank = 3? Not for: $x_1 = x_2 = 0$
 $x_3 = x_4 = 0$
 $x_1 = x_3 = 0$
 $x_2 = x_4 = 0$

If $f(A) = \mathbb{1} \Rightarrow J_{k \circ f \circ h^{-1}}(h(A))$ has rank 3 $\Rightarrow \mathbb{1}$ regular value

$\Rightarrow O(d, \mathbb{R})$ is a submanifold of dimension $d^2 - \frac{d(d+1)}{2} = \frac{d(d-1)}{2}$