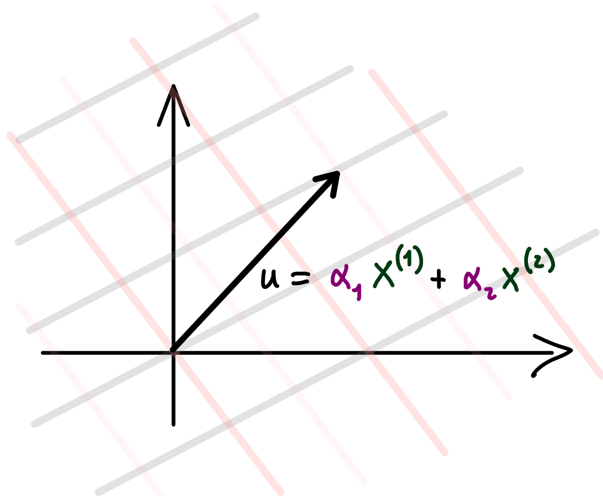
Linear Algebra - Part 65

$$\begin{tikzcd}
\text{canonical basis:} & u \in \mathbb{C}^n \arrow[rr, bend left, "A"] \arrow[d, bend left, "X^{-1}"] & & Au \in \mathbb{C}^n \arrow[d, bend left, "X^{-1}"] \\
\text{eigenvector basis:} & \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \arrow[u, bend left, "X"] \arrow[rr, bend left, "D"] & & \begin{pmatrix} \lambda_1 \alpha_1 \\ \lambda_2 \alpha_2 \\ \vdots \\ \lambda_n \alpha_n \end{pmatrix} \arrow[u, bend left, "X"]
\end{tikzcd}$$



$$\boxed{D = X^{-1} A X}$$

Is that possible? For given matrix $A \in \mathbb{C}^{n \times n}$ with eigenvectors $x^{(1)}, x^{(2)}, \dots, x^{(n)}$:

- Can we express each $u \in \mathbb{C}^n$ with $\alpha_1 x^{(1)} + \alpha_2 x^{(2)} + \dots + \alpha_n x^{(n)}$?
- $\mathrm{Span}(x^{(1)}, x^{(2)}, \dots, x^{(n)}) = \mathbb{C}^n$?
- $(x^{(1)}, x^{(2)}, \dots, x^{(n)})$ basis of $\mathbb{C}^n$?
- $X = \begin{pmatrix} | & | & & | \\ x^{(1)} & x^{(2)} & \dots & x^{(n)} \\ | & | & & | \end{pmatrix}$ invertible ?

Definition: $A \in \mathbb{C}^{n \times n}$ is called diagonalizable if one can find n eigenvectors of A such that they form a basis $\mathbb{C}^n$.

Example: (a) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, e_1, e_2 eigenvectors $\Rightarrow$ A is diagonalizable

(b) $B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvectors $\Rightarrow$ B is diagonalizable

(c) $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, all eigenvectors lie in direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\Rightarrow$ C is not diagonalizable

Remember: For $A \in \mathbb{C}^{n \times n}$:

- $\alpha(\lambda) = \gamma(\lambda)$ for all eigenvalues λ $\Leftrightarrow$ A is diagonalizable
- A normal $\Rightarrow$ A is diagonalizable
  (One can choose even an ONB with eigenvectors)
- A has n different eigenvalues $\Rightarrow$ A is diagonalizable