

Linear Algebra - Part 44

$A \in \mathbb{R}^{2 \times 2} \rightsquigarrow$ system of linear equations $Ax = b$

Assume $\neq 0$

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array} \right) \xrightarrow{\mathbb{I} - \frac{a_{21}}{a_{11}} \mathbb{I}} \left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & b_2 - \frac{a_{21}}{a_{11}} b_1 \end{array} \right) \xrightarrow{\mathbb{I} \cdot a_{11}}$$

$$\xrightarrow{\quad} \left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ 0 & a_{11}a_{22} - a_{21}a_{12} & a_{11}b_2 - a_{21}b_1 \end{array} \right)$$

$\neq 0 \iff$ we have a unique solution

Definition: For a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2}$, the number

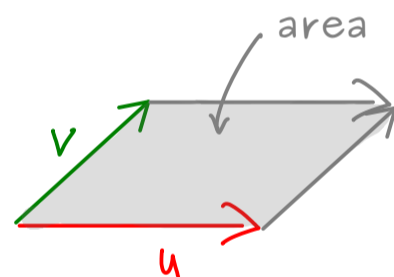
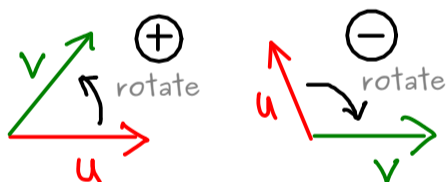
$$\det(A) := a_{11}a_{22} - a_{12}a_{21}$$

is called the determinant of A.

What about volumes? \rightsquigarrow vol_n

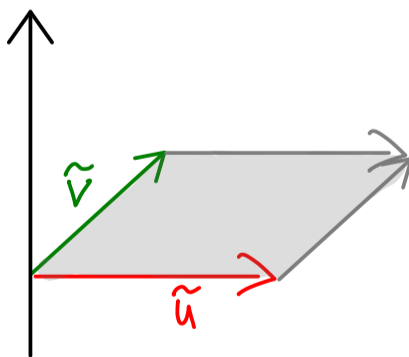
in \mathbb{R}^2 : $\text{vol}_2(u, v) :=$ orientated area of parallelogram

$\stackrel{\pm}{\parallel}$



Relation to cross product: embed \mathbb{R}^2 into \mathbb{R}^3 : $\tilde{u} := \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$, $\tilde{v} = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$

\mathbb{R}^3 :



$$\|\tilde{u} \times \tilde{v}\| = \left\| \begin{pmatrix} 0 \\ 0 \\ u_1v_2 - v_1u_2 \end{pmatrix} \right\| = \underbrace{|u_1v_2 - v_1u_2|}_{\det \begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}}$$

Result: $\text{vol}_2(u, v) = \det \begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}$ (volume function = determinant)