The Bright Side of Mathematics

Linear Algebra – Part 41

$A \in \mathbb{R}^{m \times n} \xrightarrow{\text{Gaussian elimination}}$ row echelon form

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left(\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{2} & -1 & 4 & 0 \\ 0 & 0 & 0 & \boxed{4} & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad \mathrm{Ker}(A) = \left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_5 \in \mathbb{R} \right\}$$

Remember:

$\dim(\mathrm{Ker}(A)) =$ number of free variables

$+$

$\dim(\mathrm{Ran}(A)) =$ number of leading variables

$= n$

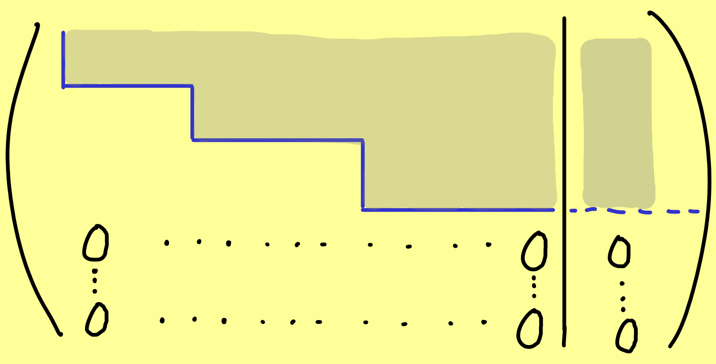
Proposition: For $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, we have the following equivalences:

(1) $Ax = b$ has at least one solution.

(2) $b \in \mathrm{Ran}(A)$

(3) b can be written as a linear combination of the columns of A.

(4) Row echelon form looks like:



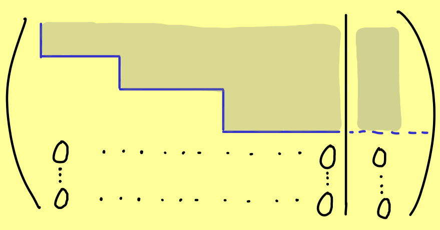
Proof: (1) $\Leftrightarrow$ (2) given by definition of $\mathrm{Ran}(A)$

(2) $\Leftrightarrow$ (3) given by column picture of $\mathrm{Ran}(A)$

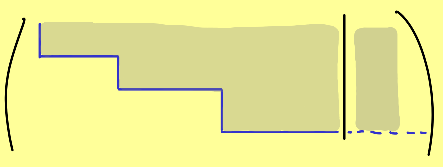
$$\mathrm{Ran}(A) = \left\{ \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix} x \;\middle|\; x \in \mathbb{R}^n \right\}$$
$$= \left\{ x_1 \cdot \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} + \cdots + x_n \begin{pmatrix} | \\ a_n \\ | \end{pmatrix} \;\middle|\; x \in \mathbb{R}^n \right\}$$

(4) $\Rightarrow$ (1)

Assume we have this:



Then solve

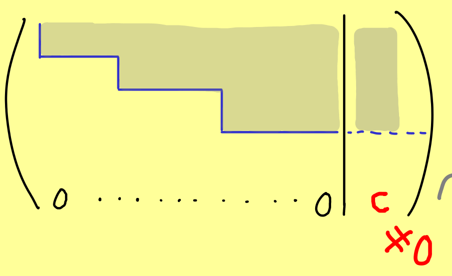


by backwards substitution.

(or argue with $\mathrm{rank}(A) = \mathrm{rank}((A|b))$)

(1) $\Rightarrow$ (4) (let's show: $\neg(4) \Rightarrow \neg(1)$)

Assume:



not solvable $\quad 0 = c \;$ ↯

$\Rightarrow$ no solution for $Ax = b$ $\quad \square$