BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Linear Algebra – Part 20

Linear map: $f: \mathbb{R}^n \to \mathbb{R}^m, \ x \mapsto f(x)$

n components

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

canonical unit vectors

$$f(x) = f(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \overset{\text{linearity}}{=} x_1 f(e_1) + x_2 f(e_2) + \cdots + x_n f(e_n)$$

$\Rightarrow$ to know $f(x)$, it's sufficient to know $f(e_1), \ldots, f(e_n)$

Proposition: $f: \mathbb{R}^n \to \mathbb{R}^m$ linear.

Then there is exactly one matrix $A \in \mathbb{R}^{m \times n}$ with $f = f_A$ $\big(f(x) = Ax \big)$

and

$$A = \begin{pmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{pmatrix}.$$

Proof:

$$f_A(x) = f_A\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} | \\ f(e_1) \\ | \end{pmatrix} + \cdots + x_n \begin{pmatrix} | \\ f(e_n) \\ | \end{pmatrix}$$

$$= f(x)$$

Uniqueness: Assume there are $A, B \in \mathbb{R}^{m \times n}$ with $f = f_A$ and $f = f_B$

$\Rightarrow$ $Ax = Bx$ for all $x \in \mathbb{R}^n$

$\Rightarrow$ $(A - B)x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ for all $x \in \mathbb{R}^n$

$\overset{\text{Use } e_i}{\Rightarrow}$ $A - B = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ $\Rightarrow$ $A = B$ $\square$