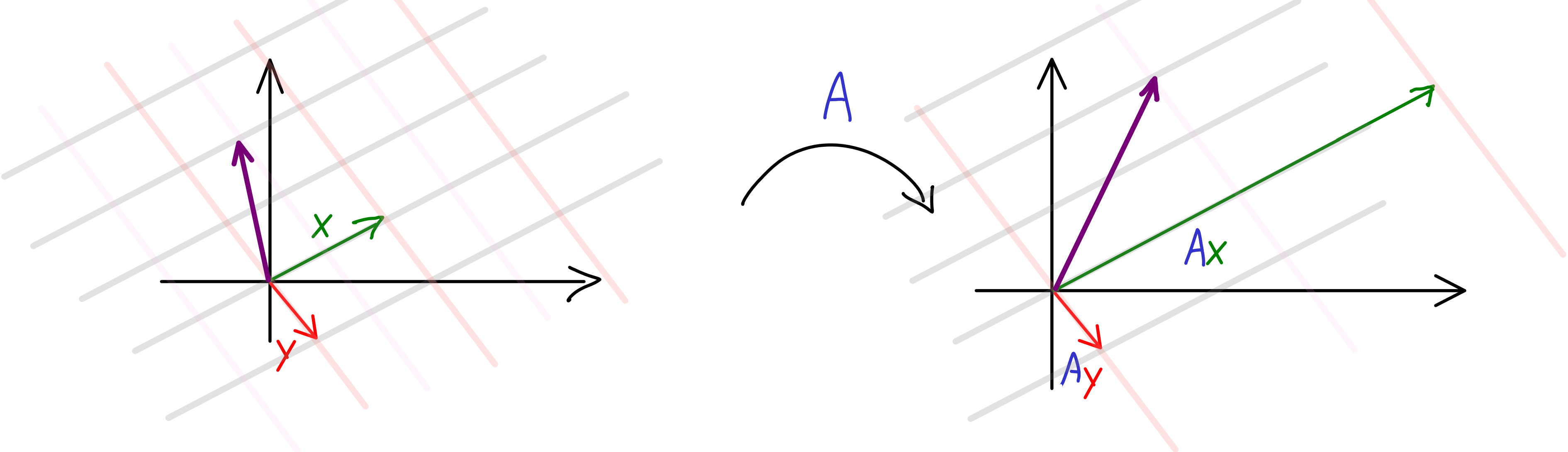
Linear Algebra – Part 64

Diagonalization = transform matrix into a diagonal one

= find a an optimal coordinate system

Example: $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$, $\lambda_1 = 4$, $\lambda_2 = 1$ (eigenvalues)

$x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (eigenvectors)



$$\alpha x + \beta y \longmapsto \alpha \lambda_1 x + \beta \lambda_2 y$$

Diagonalization: $A \in \mathbb{C}^{n \times n} \rightsquigarrow \lambda_1, \lambda_2, \dots, \lambda_n$ (counted with algebraic multiplicities)

$\rightsquigarrow x^{(1)}, x^{(2)}, \dots, x^{(n)}$ (associated eigenvectors)

$\rightsquigarrow A x^{(1)} = \lambda_1 x^{(1)}, \dots, A x^{(n)} = \lambda_n x^{(n)}$ (eigenvalue equations)

$$A \begin{pmatrix} | & | & & | \\ x^{(1)} & x^{(2)} & \cdots & x^{(n)} \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ Ax^{(1)} & Ax^{(2)} & \cdots & Ax^{(n)} \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & & | \\ \lambda_1 x^{(1)} & \lambda_2 x^{(2)} & \cdots & \lambda_n x^{(n)} \\ | & | & & | \end{pmatrix} = \underbrace{\begin{pmatrix} | & | & & | \\ x^{(1)} & x^{(2)} & \cdots & x^{(n)} \\ | & | & & | \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}}_{D}$$

$$\Longrightarrow \quad AX = XD$$

If X is invertible, then: $\boxed{D = X^{-1}AX}$ A is similar to a diagonal matrix

Application: $$A^{98} = \left(XDX^{-1}\right)^{98} = XD\underbrace{X^{-1}X}_{\mathbb{1}}D\underbrace{X^{-1}X}_{\mathbb{1}}DX^{-1}\cdots XDX^{-1} \quad \cdots$$

$$= XD^{98}X^{-1}$$

$$= X\begin{pmatrix} \lambda_1^{98} & & & \\ & \lambda_2^{98} & & \\ & & \ddots & \\ & & & \lambda_n^{98} \end{pmatrix}X^{-1}$$