



## Linear Algebra - Part 63

Assume:  $x$  eigenvector for  $A \in \mathbb{C}^{n \times n}$  associated to eigenvalue  $\lambda \in \mathbb{C}$

Then:  $Ax = \lambda x \implies A(Ax) = A(\lambda x) = \lambda (Ax)$   
 $\implies A^2 x = \lambda^2 x$  (with  $Ax = \lambda x$  substituted)

$$\implies A^2 x = \lambda^2 x \implies A^3 x = \lambda^3 x$$

induction

$$\implies A^m x = \lambda^m x \quad \text{for all } m \in \mathbb{N}$$

Spectral mapping theorem:  $A \in \mathbb{C}^{n \times n}$ ,  $p: \mathbb{C} \rightarrow \mathbb{C}$ ,  $p(z) = c_m z^m + \dots + c_1 z^1 + c_0$

Define:  $p(A) = c_m A^m + c_{m-1} A^{m-1} + \dots + c_1 A + c_0 \mathbb{1}_n \in \mathbb{C}^{n \times n}$

Then:  $\text{spec}(p(A)) = \{ p(\lambda) \mid \lambda \in \text{spec}(A) \}$

Proof: Show two inclusion:  $(\supseteq)$  (see above)  $\checkmark$

$(\subseteq)$  **1st case:**  $p$  constant,  $p(z) = c_0$ .

$$\text{Take } \tilde{\lambda} \in \text{spec}(p(A)) \implies \det(p(A) - \tilde{\lambda} \mathbb{1}) = 0$$

$(c_0 - \tilde{\lambda})^n$        $c_0 \mathbb{1}$

$$\implies \tilde{\lambda} \in \{ p(\lambda) \mid \lambda \in \text{spec}(A) \} \quad \checkmark$$

**2nd case:**  $p$  not constant. Do proof by contraposition.

Assume:  $\mu \notin \{p(\lambda) \mid \lambda \in \text{spec}(A)\}$

Define polynomial:  $q(z) = p(z) - \mu$   
 $= c \cdot (z - a_1)(z - a_2) \dots (z - a_m)$   
\*0

By definition of  $\mu$ :  $a_j \notin \text{spec}(A)$  for all  $j$

$$\Rightarrow \det(A - a_j \mathbb{1}) \neq 0 \quad \text{for all } j$$

Hence:  $\det(p(A) - \mu \mathbb{1}) = \det(q(A))$

$$= \det(c \cdot (A - a_1)(A - a_2) \dots (A - a_m))$$

$$= c^n \cdot \det(A - a_1) \det(A - a_2) \dots \det(A - a_m)$$

$$\neq 0$$

$$\Rightarrow \mu \notin \text{spec}(p(A)) \quad \square$$

Example:

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}, \quad \text{spec}(A) = \{1, 4\}$$

$$B = 3A^3 - 7A^2 + A - 2\mathbb{1}, \quad \text{spec}(B) = \{-5, 82\}$$