Linear Algebra - Part 63

Assume: x eigenvector for $A \in \mathbb{C}^{n\times n}$ associated to eigenvalue $\lambda \in \mathbb{C}$

Then: $Ax = \lambda x \implies \underbrace{A(Ax)}_{= A^2 x} = A(\lambda x) = \lambda (\underbrace{Ax}_{= \lambda x})$

$\implies A^2 x = \lambda^2 x \implies A^3 x = \lambda^3 x$

$\overset{\text{induction}}{\implies} A^m x = \lambda^m x$ for all $m \in \mathbb{N}$

Spectral mapping theorem: $A \in \mathbb{C}^{n\times n}$, $p : \mathbb{C} \to \mathbb{C}$, $p(z) = c_m z^m + \dots + c_1 z^1 + c_0$

Define: $p(A) = c_m A^m + c_{m-1} A^{m-1} + \dots + c_1 A + c_0 \cdot \mathbb{1}_n \in \mathbb{C}^{n\times n}$

Then: $\operatorname{spec}(p(A)) = \{ p(\lambda) \mid \lambda \in \operatorname{spec}(A) \}$

Proof: Show two inclusion: $(\supseteq)$ (see above) ✓

$(\subseteq)$ $\boxed{\text{1st case:}}$ p constant, $p(z) = c_0$.

Take $\tilde{\lambda} \in \operatorname{spec}(p(A)) \implies \underbrace{\det(\underbrace{p(A)}_{c_0 \mathbb{1}} - \tilde{\lambda}\mathbb{1})}_{= (c_0 - \tilde{\lambda})^n} = 0$

$\implies \tilde{\lambda} \in \{ p(\lambda) \mid \lambda \in \operatorname{spec}(A) \}$ ✓

$\boxed{\text{2nd case:}}$ p not constant. Do proof by contraposition.

Assume: $\mu \notin \{ p(\lambda) \mid \lambda \in \operatorname{spec}(A) \}$

Define polynomial:
$$q(z) = p(z) - \mu$$
$$= \underset{\neq 0}{c} \cdot (z - a_1)(z - a_2) \cdots (z - a_m)$$

By definition of μ: $\quad a_j \notin \operatorname{spec}(A) \quad$ for all j

$$\Longrightarrow \quad \det(A - a_j \mathbb{1}) \neq 0 \qquad \text{for all } j$$

Hence:
$$\det(p(A) - \mu \mathbb{1}) = \det(q(A))$$
$$= \det\big(c \cdot (A - a_1)(A - a_2) \cdots (A - a_m)\big)$$
$$= c^n \cdot \det(A - a_1) \det(A - a_2) \cdots \det(A - a_m)$$
$$\neq 0$$

$$\Longrightarrow \quad \mu \notin \operatorname{spec}(p(A)) \qquad \square$$

<u>Example:</u> $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}, \quad \operatorname{spec}(A) = \{1, 4\}$

$B = 3A^3 - 7A^2 + A - 2\mathbb{1}, \quad \operatorname{spec}(B) = \{-5, 82\}$