Linear Algebra – Part 62

Recall: $\alpha(\lambda)$ algebraic multiplicity

$\gamma(\lambda)$ geometric multiplicity (= dimension of $\mathrm{Eig}(\lambda)$)

Recipe: $A \in \mathbb{C}^{n \times n}$: (1) Calculate the zeros of $p_A(\lambda) = \det(A - \lambda \mathbb{1})$.

Call them $\lambda_1, \ldots, \lambda_k$,

with $\underbrace{\alpha(\lambda_1), \ldots, \alpha(\lambda_k)}_{\text{sum is equal to } n}$.

[$A \in \mathbb{R}^{n \times n}$, λ_j zero of p_A $\Rightarrow$ $\overline{\lambda_j}$ zero of p_A]

(2) For $j \in \{1, \ldots, k\}$: Solve LES $(A - \lambda_j \mathbb{1})x = 0$

Solution set: $\mathrm{Eig}(\lambda_j)$ (eigenspace)

(3) All eigenvectors: $\bigcup_{j=1}^{k} \mathrm{Eig}(\lambda_j) \setminus \{0\}$

Example: $A = \begin{pmatrix} 8 & 8 & 4 \\ -1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix}$

(1) $p_A(\lambda) = \det \begin{pmatrix} 8-\lambda & 8 & 4 \\ -1 & 2-\lambda & 1 \\ -2 & -4 & -2-\lambda \end{pmatrix}$

Sarrus

$$= (8-\lambda)(2-\lambda)(-2-\lambda) + 16 - 16 + 8(2-\lambda) + 4(8-\lambda) + 8(-2-\lambda)$$

$$= (8-\lambda)(-4+\lambda^2) + 16 - 8\lambda + 32 - 4\lambda - 16 - 8\lambda$$

$$= (8-\lambda)(-4+\lambda^2) - 20\lambda + 32$$

$$= -32 + 4\lambda + 8\lambda^2 - \lambda^3 - 20\lambda + 32$$

$$= \lambda(-\lambda^2 + 8\lambda - 16) = -\lambda(\lambda-4)^2$$

$p_A(\lambda) = -\lambda^1(\lambda-4)^2$

eigenvalues:

$\lambda_1 = 0$, $\alpha(\lambda_1) = 1$

$\lambda_2 = 4$, $\alpha(\lambda_2) = 2$

(2) eigenspace for $\lambda_1 = 0$

$$\mathrm{Eig}(\lambda_1) = \mathrm{Ker}\,(A - \lambda_1 \mathbb{1}) = \mathrm{Ker}\begin{pmatrix} 8 & 8 & 4 \\ -1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix} \overset{\mathrm{I} \leftrightarrow \mathrm{II}}{=} \mathrm{Ker}\begin{pmatrix} -1 & 2 & 1 \\ 8 & 8 & 4 \\ -2 & -4 & -2 \end{pmatrix}$$

$$\overset{\substack{\mathrm{II}+8\mathrm{I} \\ \mathrm{III}-2\mathrm{I}}}{=} \mathrm{Ker}\begin{pmatrix} -1 & 2 & 1 \\ 0 & 24 & 12 \\ 0 & -8 & -4 \end{pmatrix} \overset{\substack{\mathrm{II}\cdot\frac{1}{12} \\ \mathrm{III}\cdot\frac{1}{4}}}{=} \mathrm{Ker}\begin{pmatrix} -1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{pmatrix}$$

$$\overset{\mathrm{III}+\mathrm{II}}{=} \mathrm{Ker}\begin{pmatrix} -1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ -\frac{1}{2}t \\ t \end{pmatrix} \,\middle|\, t \in \mathbb{C} \right\} = \mathrm{Span}\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

eigenspace for $\lambda_2 = 4$

$$\mathrm{Eig}(\lambda_2) = \mathrm{Ker}\,(A - \lambda_2 \mathbb{1}) = \mathrm{Ker}\begin{pmatrix} 4 & 8 & 4 \\ -1 & -2 & 1 \\ -2 & -4 & -6 \end{pmatrix} \overset{\mathrm{I} \leftrightarrow \mathrm{II}}{=} \mathrm{Ker}\begin{pmatrix} -1 & -2 & 1 \\ 4 & 8 & 4 \\ -2 & -4 & -6 \end{pmatrix}$$

$$\overset{\substack{\mathrm{II}+4\mathrm{I} \\ \mathrm{III}-2\mathrm{I}}}{=} \mathrm{Ker}\begin{pmatrix} -1 & -2 & 1 \\ 0 & 0 & 8 \\ 0 & 0 & -8 \end{pmatrix} \overset{\mathrm{III}+\mathrm{II}}{=} \mathrm{Ker}\begin{pmatrix} -1 & -2 & 1 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\mathrm{II}\cdot\frac{1}{8}}{=} \mathrm{Ker}\begin{pmatrix} -1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

(3) eigenvectors of A:

$$\left(\mathrm{Span}\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \cup \mathrm{Span}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right) \setminus \{0\}$$