



## Linear Algebra - Part 62

Recall:  $\alpha(\lambda)$  algebraic multiplicity

$\gamma(\lambda)$  geometric multiplicity (= dimension of  $\text{Eig}(\lambda)$ )

Recipe:  $A \in \mathbb{C}^{n \times n}$ : (1) Calculate the zeros of  $p_A(\lambda) = \det(A - \lambda \mathbb{1})$ .

Call them  $\lambda_1, \dots, \lambda_k$ ,  
with  $\alpha(\lambda_1), \dots, \alpha(\lambda_k)$ .  
sum is equal to  $n$

$$\left[ A \in \mathbb{R}^{n \times n}, \lambda_j \text{ zero of } p_A \Rightarrow \overline{\lambda_j} \text{ zero of } p_A \right]$$

(2) For  $j \in \{1, \dots, k\}$ : solve LES  $(A - \lambda_j \mathbb{1})x = 0$

solution set:  $\text{Eig}(\lambda_j)$  (eigenspace)

(3) All eigenvectors:  $\bigcup_{j=1}^k \text{Eig}(\lambda_j) \setminus \{0\}$

Example:

$$A = \begin{pmatrix} 8 & 8 & 4 \\ -1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix}$$

$$(1) p_A(\lambda) = \det \begin{pmatrix} 8-\lambda & 8 & 4 \\ -1 & 2-\lambda & 1 \\ -2 & -4 & -2-\lambda \end{pmatrix}$$

$$p_A(\lambda) = -\lambda^1(\lambda-4)^2$$

eigenvalues:

$$\lambda_1 = 0, \alpha(\lambda_1) = 1$$

$$\lambda_2 = 4, \alpha(\lambda_2) = 2$$

Sarrus

$$= (8-\lambda)(2-\lambda)(-2-\lambda) + 16 - 16 + 8(2-\lambda) + 4(8-\lambda) + 8(-2-\lambda)$$

$$= (8-\lambda)(-4+\lambda^2) + 16 - 8\lambda + 32 - 4\lambda - 16 - 8\lambda$$

$$= (8-\lambda)(-4+\lambda^2) - 20\lambda + 32$$

$$= -32 + 4\lambda + 8\lambda^2 - \lambda^3 - 20\lambda + 32$$

$$= \lambda(-\lambda^2 + 8\lambda - 16) = -\lambda(\lambda-4)^2$$

(2) eigenspace for  $\lambda_1 = 0$

$$\text{Eig}(\lambda_1) = \text{Ker}(A - \lambda_1 \mathbb{1}) = \text{Ker} \begin{pmatrix} 8 & 8 & 4 \\ -1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix} \stackrel{\text{I} \leftrightarrow \text{II}}{=} \text{Ker} \begin{pmatrix} -1 & 2 & 1 \\ 8 & 8 & 4 \\ -2 & -4 & -2 \end{pmatrix}$$

$$\stackrel{\text{II}+8\text{I}}{\text{III}-2\text{I}}{=} \text{Ker} \begin{pmatrix} -1 & 2 & 1 \\ 0 & 24 & 12 \\ 0 & -8 & -4 \end{pmatrix} \stackrel{\text{II} \cdot \frac{1}{12}}{\text{III} \cdot \frac{1}{4}}{=} \text{Ker} \begin{pmatrix} -1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{pmatrix}$$

$$\stackrel{\text{III}+\text{II}}{=} \text{Ker} \begin{pmatrix} -1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ -\frac{1}{2}t \\ t \end{pmatrix} \mid t \in \mathbb{C} \right\} = \text{span} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

eigenspace for  $\lambda_2 = 4$

$$\text{Eig}(\lambda_2) = \text{Ker}(A - \lambda_2 \mathbb{1}) = \text{Ker} \begin{pmatrix} 4 & 8 & 4 \\ -1 & -2 & 1 \\ -2 & -4 & -6 \end{pmatrix} \stackrel{\text{I} \leftrightarrow \text{II}}{=} \text{Ker} \begin{pmatrix} -1 & -2 & 1 \\ 4 & 8 & 4 \\ -2 & -4 & -6 \end{pmatrix}$$

$$\stackrel{\text{II}+4\text{I}}{\text{III}-2\text{I}}{=} \text{Ker} \begin{pmatrix} -1 & -2 & 1 \\ 0 & 0 & 8 \\ 0 & 0 & -8 \end{pmatrix} \stackrel{\text{III}+\text{II}}{=} \text{Ker} \begin{pmatrix} -1 & -2 & 1 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\stackrel{\text{I} \cdot \frac{1}{8}}{=} \text{Ker} \begin{pmatrix} -1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{span} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

(3) eigenvectors of  $A$ :  $\left( \text{span} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \cup \text{span} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right) \setminus \{0\}$