Linear Algebra – Part 60

Definition: A complex matrix $A \in \mathbb{C}^{n \times n}$ is called:

(1) selfadjoint if $A^* = A$

(2) skew-adjoint $A^* = -A$

(3) unitary if $A^*A = AA^* = \mathbb{1}$ (=identity matrix)

(4) normal if $A^*A = AA^*$

Example:

(a) $A = \begin{pmatrix} 1 & 2i \\ -2i & 0 \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} \overline{1} & \overline{-2i} \\ \overline{2i} & \overline{0} \end{pmatrix} = \begin{pmatrix} 1 & 2i \\ -2i & 0 \end{pmatrix} = A$

(b) $A = \begin{pmatrix} i & -1+2i \\ 1+2i & 3i \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} \overline{i} & \overline{1+2i} \\ \overline{-1+2i} & \overline{3i} \end{pmatrix} = \begin{pmatrix} -i & 1-2i \\ -1-2i & -3i \end{pmatrix} = -A$

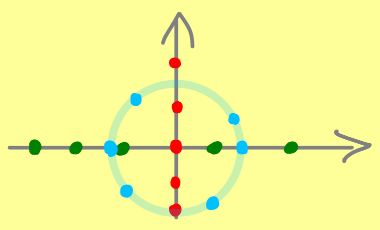
(c) $A = \begin{pmatrix} i & 0 \\ 0 & 4 \end{pmatrix}$ not selfadoint nor skew-adjoint but normal.

Remember:

$A \in \mathbb{C}^{n \times n}$	$A \in \mathbb{R}^{n \times n}$
adjoint A^*	transpose A^T
selfadjoint	symmetric
skew-adjoint	skew-symmetric
unitary	orthogonal

Proposition:

(a) A selfadjoint $\Rightarrow$ $\mathrm{spec}(A) \subseteq$ real axis



(b) A skew-adjoint $\Rightarrow$ $\mathrm{spec}(A) \subseteq$ imaginary axis

(c) A unitary $\Rightarrow$ $\mathrm{spec}(A) \subseteq$ unit circle

Proof:

(a) $\lambda \in \mathrm{spec}(A) \Rightarrow$ eigenvalue equation $Ax = \lambda x$, $x \neq 0$, choose: $\|x\| = 1$

$$\lambda \cdot \underbrace{\langle x, x \rangle}_{1} = \langle x, \lambda \cdot x \rangle = \langle x, Ax \rangle = \langle A^* x, x \rangle \overset{\text{selfadjoint}}{=} \langle Ax, x \rangle = \langle \lambda \cdot x, x \rangle = \overline{\lambda} \underbrace{\langle x, x \rangle}_{=1}$$

(c) $\lambda \in \mathrm{spec}(A) \Rightarrow$ eigenvalue equation $Ax = \lambda x$, $x \neq 0$, choose: $\|x\| = 1$

$$\langle \lambda x, \lambda x \rangle = \langle Ax, Ax \rangle = \langle \underbrace{A^* A}_{\mathbb{1}} x, x \rangle = \langle x, x \rangle = 1$$

$$\langle \lambda x, \lambda x \rangle = \overline{\lambda} \cdot \lambda \langle x, x \rangle = |\lambda|^2 \Rightarrow \lambda \text{ lies on the unit circle} \quad \square$$