

Linear Algebra - Part 59

Recall: in \mathbb{R}^n : $\langle x, y \rangle = \sum_{k=1}^n x_k y_k$

in \mathbb{C}^n : $\langle x, y \rangle = \sum_{k=1}^n \bar{x}_k y_k$

in \mathbb{R}^n : $\langle x, Ay \rangle = \langle A^T x, y \rangle$

$$\sum_{k=1}^n x_k (Ay)_k = \sum_{k=1}^n x_k a_{kj} y_j = \sum_{k=1}^n (A^T)_{jk} x_k y_j$$

in \mathbb{C}^n : $\langle x, Ay \rangle = \sum_{k=1}^n \bar{x}_k a_{kj} y_j = \sum_{k=1}^n a_{kj} \bar{x}_k y_j = \sum_{k=1}^n \overline{(A^T)_{jk} x_k} y_j$

$$= \langle A^* x, y \rangle$$

Definition: For $A \in \mathbb{C}^{m \times n}$ with $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & a_{mn} \end{pmatrix}$,

$$A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & \dots & \overline{a_{m1}} \\ \overline{a_{12}} & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \vdots \\ \overline{a_{1n}} & \dots & \dots & \overline{a_{mn}} \end{pmatrix} \in \mathbb{C}^{n \times m}$$

is called the adjoint matrix/ conjugate transpose/ Hermitian conjugate.

Examples: (a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

(b) $A = \begin{pmatrix} i & 1+i & 0 \\ 2 & e^i & 1-i \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} -i & 2 \\ 1-i & e^i \\ 0 & 1+i \end{pmatrix}$

Remember: in \mathbb{R}^n : $\langle x, y \rangle = x^T y$ (standard inner product)

in \mathbb{C}^n : $\langle x, y \rangle = x^* y$ (standard inner product)

Proposition: $\text{spec}(A^*) = \{ \bar{\lambda} \mid \lambda \in \text{spec}(A) \}$

