The Bright Side of Mathematics

Linear Algebra – Part 56

eigenvalues: $\lambda \in \operatorname{spec}(A) \iff \underbrace{\det(A - \lambda \mathbb{1})}_{\text{characteristic polynomial}} = 0$

Next step for a given $\lambda \in \operatorname{spec}(A)$:

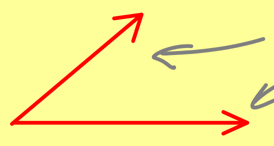
$$\operatorname{Ker}(A - \lambda \mathbb{1}) \supsetneq \{0\}$$

Solve: $\left(\begin{array}{cccc|c} a_{11}-\lambda & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22}-\lambda & & \vdots & 0 \\ \vdots & & \ddots & & \vdots \\ a_{n1} & \cdots & & a_{nn}-\lambda & 0 \end{array}\right)$

Solution set: eigenspace (associated to λ)

Definition: $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{R}$ eigenvalue

$$\gamma(\lambda) := \dim(\operatorname{Ker}(A - \lambda \mathbb{1}))$$ geometric multiplicity of λ



eigenvectors span eigenspace

Example: $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ characteristic polynomial:

$$\det(A - \lambda \mathbb{1}) = (2-\lambda)(2-\lambda)(3-\lambda) = (2-\lambda)^2 (3-\lambda)$$

$$\implies \operatorname{spec}(A) = \{2, 3\}$$

algebraic multiplicity 2   algebraic multiplicity 1

$$\operatorname{Ker}(A - 2 \cdot \mathbb{1}) = \operatorname{Ker} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

solve system: $\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{exchange II and III}} \left(\begin{array}{ccc|c} 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\rightsquigarrow x_2 = 0$, $\rightsquigarrow x_3 = 0$

x_1 free variable

backwards substitution

solution set: $\left\{ \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} \,\middle|\, x_1 \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)$

eigenvector

$$\implies \text{geometric multiplicity } \gamma(2) = 1 < \alpha(2)$$