Linear Algebra - Part 55

$$\lambda \in \operatorname{spec}(A) \iff \det(A - \lambda \mathbb{1}) = 0$$

Fundamental theorem of algebra: For $a_n \neq 0$ and $a_n, a_{n-1}, \dots, a_0 \in \mathbb{C}$, we have:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

has n solutions $x_1, x_2, \dots, x_n \in \mathbb{C}$ (not necessarily distinct).

Hence: $p(x) = a_n (x - x_n) \cdot (x - x_{n-1}) \cdots (x - x_1)$

Conclusion for characteristic polynomial: $A \in \mathbb{R}^{n \times n}$, $p_A(\lambda) := \det(A - \lambda \mathbb{1})$

- $p_A(\lambda) = 0$ has at least one solution in $\mathbb{C}$

  $\Rightarrow$ A has at least one eigenvalue in $\mathbb{C}$

  Example: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow p_A(\lambda) = \lambda^2 + 1$

  $\Rightarrow$ $-i$ and i are eigenvalues

- $p_A(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$

  Example: $A = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 1 & \\ & & & 2 \end{pmatrix} \Rightarrow p_A(\lambda) = (\lambda - 1)^2 (\lambda - 2)^2$

Definition: If $\tilde{\lambda}$ occurs k times in the factorisation $p_A(\lambda) = (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$,

then we say: $\tilde{\lambda}$ has algebraic multiplicity $k =: \alpha(\tilde{\lambda})$

Remember:
- If $\tilde{\lambda} \in \operatorname{spec}(A) \iff 1 \leq \alpha(\tilde{\lambda}) \leq n$
- $\sum_{\tilde{\lambda} \in \mathbb{C}} \alpha(\tilde{\lambda}) = n$