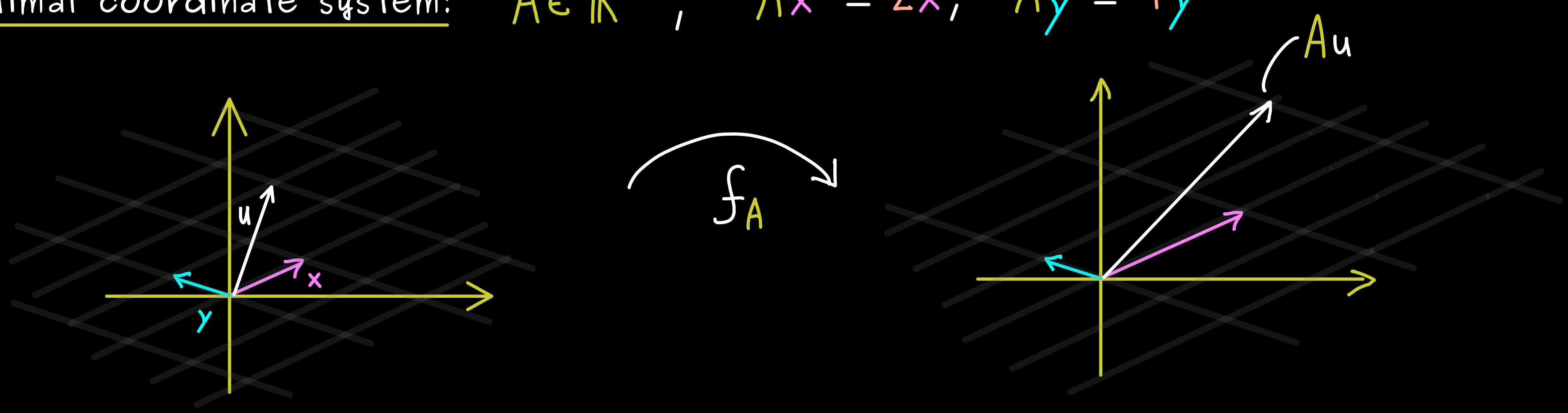
Linear Algebra - Part 54

$A \in \mathbb{R}^{n \times n} \quad \leftrightsquigarrow \quad f_A : \mathbb{R}^n \to \mathbb{R}^n$ linear map

eigenvalue equation: $Ax = \lambda \cdot x \;, \quad x \neq 0$

optimal coordinate system: $A \in \mathbb{R}^{2\times 2}, \quad Ax = 2x, \quad Ay = 1y$



$u = a \cdot x + b \cdot y$

$$\begin{aligned} Au &= A(a\cdot x + b \cdot y) \\ &= a \cdot Ax + b\, Ay \\ &= 2ax + 1by \end{aligned}$$

How to find enough eigenvectors?

$x \neq 0$ eigenvector associated to eigenvalue $\lambda \iff x \in \mathrm{Ker}(\underbrace{A - \lambda \mathbb{1}}_{\text{singular matrix}})$

$\det(A - \lambda \mathbb{1}) = 0 \iff \mathrm{Ker}(A - \lambda\mathbb{1})$ is non-trivial

$\iff \lambda$ is eigenvalue of A

Example: $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}, \quad A - \lambda \mathbb{1} = \begin{pmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{pmatrix}$

$$\det \begin{pmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{pmatrix} = (3-\lambda)(4-\lambda) - 2$$ characteristic polynomial

$$= 10 - 7\lambda + \lambda^2$$

$$= (\lambda - 5)(\lambda - 2) \overset{!}{=} 0$$

$\Rightarrow$ 2 and 5 are eigenvalues of A

General case: For $A \in \mathbb{R}^{n \times n}$:

$$\det(A - \lambda \mathbb{1}) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} - \lambda \end{pmatrix}$$

$$\overset{\text{Leibniz formula}}{=} (a_{11} - \lambda) \cdots (a_{nn} - \lambda) + \cdots$$

$$= (-1)^n \cdot \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda^1 + c_0$$

Definition: For $A \in \mathbb{R}^{n \times n}$, the polynomial of degree n given by

$$p_A : \lambda \longmapsto \det(A - \lambda \mathbb{1})$$

is called the characteristic polynomial of A.

Remember: The zeros of the characteristic polynomial are exactly the eigenvalues of A.