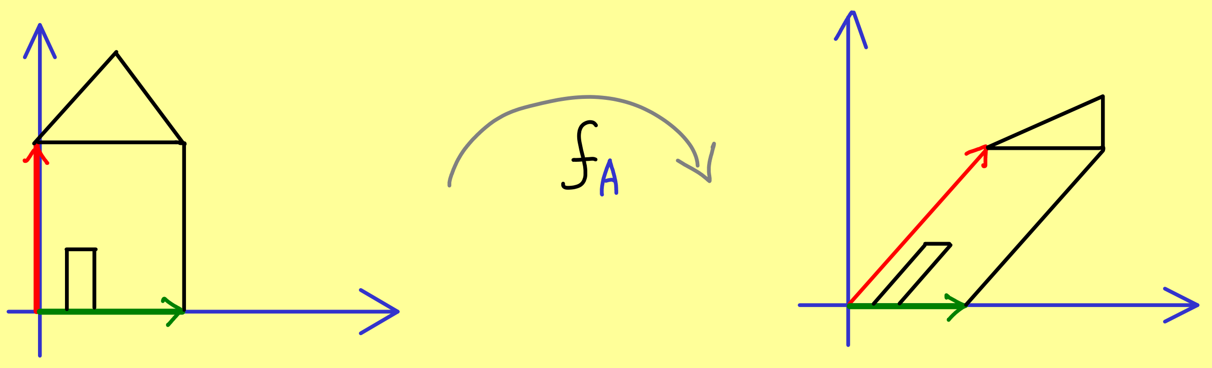
The Bright Side of Mathematics

Linear Algebra – Part 53

eigenvalue (German: Eigenwert) (David Hilbert, 1904)
↳ proper/own/characteristic

Consider: $A \in \mathbb{R}^{n \times n} \quad \leftrightsquigarrow \quad f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ linear map



Question: Are there vectors which are only scaled by f_A?

Answer: $Ax = \lambda \cdot x$ for a number $\lambda \in \mathbb{R}$

$\Longleftrightarrow (A - \lambda \mathbb{1})x = 0$ for a number $\lambda \in \mathbb{R}$

$\Longleftrightarrow x \in \mathrm{Ker}(A - \lambda \mathbb{1})$ for a number $\lambda \in \mathbb{R}$

eigenvector (if $x \neq 0$) eigenvalue

Example:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{aligned} x_1 + x_2 &= \lambda \cdot x_1 \quad \mathrm{I} \\ x_2 &= \lambda \cdot x_2 \quad \mathrm{II} \end{aligned}$$

For II: $\boxed{\lambda = 1}$ or $\underbrace{x_2 = 0}_{\mathrm{I}}$ $\Rightarrow x_1 = \lambda \cdot x_1 \Rightarrow \boxed{\lambda = 1}$ or $x_1 = 0$

For I: $x_1 + x_2 = x_1 \Rightarrow x_2 = 0$

Solution: eigenvalue: $\lambda = 1$

eigenvectors: $x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ for $x_1 \in \mathbb{R} \setminus \{0\}$

Definition: $A \in \mathbb{R}^{n \times n}, \ \lambda \in \mathbb{R}$.

If there is $x \in \mathbb{R}^n \setminus \{0\}$ with $Ax = \lambda x$, then:

- λ is called an eigenvalue of A
- x is called an eigenvector of A (associated to λ)
- $\mathrm{Ker}(A - \lambda \mathbb{1})$ eigenspace of A (associated to λ)

The set of all eigenvalues of A: $\mathrm{spec}(A)$ spectrum of A